Exercise 5.11

(a) Let $Y = a_{\min(K,n)}$. As an aside, note that $Y$ is the present value random variable associated with an $n$-year life annuity-immediate on $(x)$. Rewriting $Y$ as

$$Y = \frac{1 - v^{\min(K,n)}}{i} \cdot \frac{v - v^{\min(K+1,n+1)}}{iv} = \frac{v - v^{\min(K+1,n+1)}}{d},$$

we find

$$\text{Var}[Y] = \frac{1}{d^2} \text{Var}[v^{\min(K+1,n+1)}] = \frac{2A_{x:n+1}^1 - \left(A_{x:n+1}^1\right)^2}{d^2},$$

because $v^{\min(K+1,n+1)}$ is the present value random variable of an $(n+1)$-year endowment to $(x)$.

(b) Now, express $Y$ as

$$Y = \frac{1}{i} \left[1 - (v^K I(K < n) + v^n I(K \geq n))\right]$$

and define $Z_1 = v^K I(K < n)$ and $Z_2 = v^n I(K \geq n)$ so that clearly $Y = \frac{1}{i} [1 - (Z_1 + Z_2)]$. The variance of $Y$ therefore can be written as

$$\text{Var}[Y] = \frac{1}{i^2} \text{Var}[Z_1 + Z_2] = \frac{1}{i^2} \left\{\text{Var}[Z_1] + 2\text{Cov}[Z_1, Z_2] + \text{Var}[Z_2]\right\}. \quad (1)$$

We note that

$$\text{Var}[Z_1] = \frac{1}{v^2} \text{Var}[v^{K+1} I(K < n)] = (1 + i)^2 \left[2A_{x:m}^1 - \left(A_{x:m}^1\right)^2\right]. \quad (2)$$

Since $Z_2$ is the present value random variable of an $n$-year pure endowment and that $Z_1 Z_2 = 0$, we find

$$\text{Cov}[Z_1, Z_2] = -E[Z_1] \cdot E[Z_2] = -\frac{1}{v} A_{x:m}^1 \cdot v^n n P_x = -(1 + i)A_{x:m}^1 \cdot v^n n P_x \quad (3)$$

and that

$$\text{Var}[Z_2] = v^{2n} n P_x (1 - n P_x). \quad (4)$$

Finally, plugging the results of (2), (3) and (4) into (1), we get the desired result.