Exercise 4.6

For \((IA^{(m)})_x\), the benefit starts with $1 if death occurs in the first year and increasing by $1 each year for deaths thereafter. Obviously, the benefit is payable at the end of the \(m\)-th interval in the year of death. The benefit payment pattern is best visualize in the following picture:

This visualization allows us to interpret the Actuarial Present Value as the sum of deferred insurances each with $1 of death benefits with detailed proof as follows:

\[
(IA^{(m)})_x = \sum_{k=0}^{\infty} \sum_{j=0}^{m-1} (k + 1) v^{k+j+1} m \cdot \frac{1}{m} q_x + k
\]

\[
= \sum_{k=0}^{\infty} \sum_{j=0}^{m-1} v^{k+j+1} m \cdot \frac{1}{m} q_x + k
\]

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\]

change the order of the first two summations

\[
= \sum_{s=0}^{\infty} \left[ \sum_{k=s}^{\infty} \sum_{j=0}^{m-1} v^{k+j+1} m \cdot \frac{1}{m} q_x + k \right]
\]

\[
= \sum_{s=0}^{\infty} s! A^{(m)}_x
\]

where in the second line, we write \((k + 1) = \sum_{s=0}^{k} 1\), that is, the sum of 1’s \(k + 1\) times.
The result immediately follows because we can write

\[
(I A^{(m)}_x)_s = \sum_{s=0}^{\infty} a^{(m)}_x
\]

\[
= \sum_{s=0}^{\infty} v^s p_x A^{(m)}_{x+s}
\]

\[
= A^{(m)}_x + vp_x A^{(m)}_{x+1} + v^2 p_x A^{(m)}_{x+2} + \cdots
\]

Thus, we see that equivalently, if death occurs in the first year, benefit is $1, increasing by $1 each year of death thereafter. The extra $1 comes from each term whenever death is prolonged for another year.