Central theme: to quantify the value today of a (random) amount to be paid at a random time in the future.

- main application is in life insurance contracts, but could be applied in other contexts, e.g. warranty contracts.

Generally computed in two steps:

1. take the present value (PV) random variable, $b_T v_T$; and
2. calculate the expected value $E[b_T v_T]$ for the average value - this value is referred to as the Actuarial Present Value (APV).

In general, we want to understand the entire distribution of the PV random variable $b_T v_T$:

- it could be highly skewed, in which case, there is danger to use expectation.
- other ways of summarizing the distribution such as variances and percentiles/quantiles may be useful.
A simple illustration

Consider the simple illustration of valuing a three-year term insurance policy issued to age 35 where if he dies within the first year, a $1,000 benefit is payable at the end of his year of death.

If he dies within the second year, a $2,000 benefit is payable at the end of his year of death. If he dies within the third year, a $5,000 benefit is payable at the end of his year of death.

Assume a constant discount rate of 5% and the following extract from a mortality table:

<table>
<thead>
<tr>
<th></th>
<th>q_x</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>0.005</td>
</tr>
<tr>
<td>36</td>
<td>0.006</td>
</tr>
<tr>
<td>37</td>
<td>0.007</td>
</tr>
<tr>
<td>38</td>
<td>0.008</td>
</tr>
</tbody>
</table>

Calculate the APV of the benefits.
Chapter summary

- Life insurance
  - benefits payable contingent upon death; payment made to a designated beneficiary
  - actuarial present values (APV)
  - actuarial symbols and notation

- Insurances payable at the moment of death
  - continuous
  - level benefits, varying benefits (e.g. increasing, decreasing)

- Insurances payable at the end of year of death
  - discrete
  - level benefits, varying benefits (e.g. increasing, decreasing)

- Chapter 4 (Dickson, et al.) - both 1st/2nd ed.
The present value random variable

Denote by $Z$, the **present value** random variable.

This gives the value, at policy issue, of the benefit payment. Issue age is usually denoted by $x$.

In the case where the benefit is payable at the moment of death, $Z$ clearly depends on the time-until-death $T$. For simplicity, we drop the subscript $x$ for age-at-issue.

It is $Z = b_T v_T$ where:

- $b_T$ is called the benefit payment function
- $v_T$ is the discount function

In the case where we have a constant (fixed) interest rate, then $v_T = v^T = (1 + i)^{-T} = e^{-\delta T}$.
Fixed term life insurance

- An $n$-year term life insurance provides payment if the insured dies within $n$ years from issue.
- For a unit of benefit payment, we have
  \[ b_T = \begin{cases} 
  1, & T \leq n \\
  0, & T > n 
  \end{cases} \quad \text{and} \quad v_T = v^T. \]
- The present value random variable is therefore
  \[ Z = \begin{cases} 
  v^T, & T \leq n \\
  0, & T > n 
  \end{cases} = v^T I(T \leq n) \]
  where $I(\cdot)$ is called indicator function. $E[Z]$ is called the APV of the insurance.
- Actuarial notation:
  \[ \bar{A}_{x:n}^1 = E[Z] = \int_0^n v^t f_x(t) dt = \int_0^n v^t t p_x \mu_x + t dt. \]
Rule of moments

- The $j$-th moment of the distribution of $Z$ can be expressed as:

$$E[Z^j] = \int_0^n v^j_t p_x \mu_x + t dt = \int_0^n e^{-(j \delta_t)} t p_x \mu_x + t dt.$$  

- This is actually equal to the APV but evaluated at the force of interest $j \delta_t$.

- In general, we have the following rule of moment:

$$E[Z^j] @ \delta_t = E[Z] @ j \delta_t.$$  

- For example, the variance can be expressed as

$$\text{Var}[Z] = 2 \overline{A}^1_{x:n} - (\overline{A}^1_{x:n})^2.$$
Whole life insurance

- For a whole life insurance, benefits are payable following death at any time in the future.

- Here, we have $b_T = 1$ so that the present value random variable is $Z = v^T$.

- APV notation for whole life: $\bar{A}_x = E[Z] = \int_0^\infty v^t p_x e^{x+t} dt$.

- Variance (using rule of moments):

  $$\text{Var}[Z] = 2\bar{A}_x - (\bar{A}_x)^2.$$ 

- Whole life insurance is the limiting case of term life insurance as $n \to \infty$.

- Note also that if the benefit amount is not 1, but say $b_T = b$, then $E[Z] = b \bar{A}_x$ and that $\text{Var}[Z] = b^2 \left[2\bar{A}_x - (\bar{A}_x)^2\right]$.
Pure endowment insurance

- For an \( n \)-year pure endowment insurance, a benefit is payable at the end of \( n \) years if the insured survives at least \( n \) years from issue.

- Here, we have \( b_T = \begin{cases} 0, & T \leq n \\ 1, & T > n \end{cases} \) and \( v_T = v^n \) so that the PV r.v. is

\[
Z = \begin{cases} 0, & T \leq n \\ v^n, & T > n \end{cases}.
\]

- APV for pure endowment: \( A_{x: \frac{1}{n}} = nE_x = v^n np_x \).

- Variance (using rule of moments):

\[
\text{Var}[Z] = v^{2n} np_x \cdot nq_x = 2A_{x: \frac{1}{n}} - \left(A_{x: \frac{1}{n}}\right)^2.
\]

- Sometimes, we can also express the present value random variable based on an indicator function:

\[
Z = v^n I(T_x > n),
\]

where \( I(E) \) is 1 if the event \( E \) is true, and 0 otherwise.
Endowment insurance

- For an \( n \)-year endowment insurance, a benefit is payable if death is within \( n \) years or if the insured survives at least \( n \) years from issue, whichever occurs first.

- Here, we have \( b_T = 1 \) and \( v_T = \begin{cases} v^T, & T \leq n \\ v^n, & T > n \end{cases} \) so that the PV r.v. is

\[
Z = \begin{cases} v^T, & T \leq n \\ v^n, & T > n \end{cases}.
\]

- It is easy to see that we can re-write \( Z \) as \( Z = v^\min(T,n) \).

- APV endowment: \( \bar{A}_{x:\overline{n}} = \bar{A}_{x:1}^{1} + A_{x:1}^{1} \).

- Variance (using rule of moments):

\[
\text{Var}[Z] = 2\bar{A}_{x:\overline{n}} - (\bar{A}_{x:\overline{n}})^2.
\]
Deferred insurance

- For an $n$-year deferred whole insurance, a benefit is payable if the insured dies at least $n$ years following issue.

- Here, we have $b_T = \begin{cases} 0, & T \leq n \\ 1, & T > n \end{cases}$ and $v_T = v^T$ so that the PV r.v. is

$$Z = \begin{cases} 0, & T \leq n \\ v^T, & T > n \end{cases}.$$

- **APV for deferred insurance:**

$$n|\bar{A}_x = \int_n^\infty v^t_i P_x \mu_{x+t} dt.$$

- **Variance (using rule of moments):**

$$\text{Var}[Z] = 2n|\bar{A}_x - (n|\bar{A}_x)^2.$$
Constant force of mortality - all throughout life

Assume mortality is based on a constant force, say $\mu$, and interest is also based on a constant force of interest, say $\delta$.

- Find expressions for the APV for the following types of insurances:
  - whole life insurance;
  - $n$-year term life insurance;
  - $n$-year endowment insurance; and
  - $m$-year deferred life insurance.
- Check out the (corresponding) variances for each of these types of insurance.

[Details in class]
De Moivre’s law

Find expressions for the APV for the same types of insurances in the case where you have:

- De Moivre’s law.
Illustrative example 1

For a whole life insurance of $1,000 on \((x)\) with benefits payable at the moment of death, you are given:

\[
\delta_t = \begin{cases} 
0.04, & 0 < t \leq 10 \\
0.05, & t > 10 
\end{cases}
\]

and

\[
\mu_{x+t} = \begin{cases} 
0.006, & 0 < t \leq 10 \\
0.007, & t > 10 
\end{cases}
\]

Calculate the actuarial present value for this insurance.
Equivalent probability calculations

We can also compute probabilities of $Z$ as follows. Consider the present value random variable $Z$ for a whole life issued to age $x$. For $0 < \alpha < 1$, the following is straightforward:

\[
\Pr[Z \leq \alpha] = \Pr[e^{-\delta T_x} \leq \alpha = \Pr[-\delta T_x \leq \log(\alpha)]
= \Pr[T_x > -(1/\delta) \log(\alpha)] = u p_x,
\]

where

\[
u = (1/\delta) \log(1/\alpha) = \log(1/\alpha)^{1/\delta}.
\]

- Consider the case where $\alpha = 0.75$ and $\delta = 0.05$. Then
  \[
u = \log(1/0.75)^{1/0.05} = 5.753641.
\]
- Thus, the probability $\Pr[Z \leq 0.75]$ is equivalent to the probability that $(x)$ will survive for another 5.753641 years.
## Varying benefits

### Insurances with varying benefits

<table>
<thead>
<tr>
<th>Type</th>
<th>$b_T$</th>
<th>$Z$</th>
<th>APV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Increasing whole life</td>
<td>$[T + 1]$</td>
<td>$[T + 1]v^T$</td>
<td>$(I \bar{A})_x$</td>
</tr>
<tr>
<td>Whole life increasing $m$-thly</td>
<td>$[Tm + 1]/m$</td>
<td>$v^T[Tm + 1]/m$</td>
<td>$(I^{(m)} \bar{A})_x$</td>
</tr>
<tr>
<td>Constant increasing whole life</td>
<td>$T$</td>
<td>$Tv^T$</td>
<td>$(\bar{I} \bar{A})_x$</td>
</tr>
<tr>
<td>Decreasing $n$-year term</td>
<td>$\begin{cases} n - \lfloor T \rfloor, &amp; T \leq n \ 0, &amp; T &gt; n \end{cases}$</td>
<td>$\begin{cases} (n - \lfloor T \rfloor)v^T, &amp; T \leq n \ 0, &amp; T &gt; n \end{cases}$</td>
<td>$(D \bar{A})_{x:n}^{1}$</td>
</tr>
</tbody>
</table>

* These items will be discussed in class.
Illustrative example 2

For a whole life insurance on (50) with death benefits payable at the moment of death, you are given:

- Mortality follows De Moivre’s law with $\omega = 110$.
- $b_t = 10000(1.10)^t$, for $t \geq 0$
- $\delta = 5\%$
- $Z$ denotes the present value random variable for this insurance.

Calculate $E[Z]$ and $Var[Z]$.

Can you find an explicit expression for the distribution function of $Z$, i.e. $Pr[Z \leq z]$?
For insurances payable at the end of the year (EOY) of death, the PV r.v. $Z$ clearly depends on the curtate future lifetime $K_x$.

It is $Z = b_{K+1}v_{K+1}$.

To illustrate, consider an $n$-year term insurance which pays benefit at the end of year of death:

\[
b_{K+1} = \begin{cases} 
1, & K = 0, 1, \ldots, n-1 \\
0, & \text{otherwise} 
\end{cases}, \quad v_{K+1} = v_{K+1},
\]

and therefore

\[
Z = \begin{cases} 
v_{K+1}, & K = 0, 1, \ldots, n-1 \\
0, & \text{otherwise} 
\end{cases}.
\]
APV of $n$-year term:

$$A_{x:n}^1 = \mathbb{E}[Z] = \sum_{k=0}^{n-1} v^{k+1} q_x = \sum_{k=0}^{n-1} v^{k+1} p_x \cdot q_{x+k}$$

Rule of moments also apply in discrete situations. For example,

$$\text{Var}[Z] = 2A_{x:n}^1 - (A_{x:n}^1)^2,$$

where

$$2A_{x:n}^1 = \mathbb{E}[Z^2] = \sum_{k=0}^{n-1} e^{-2\delta(k+1)} p_x \cdot q_{x+k}.$$
(Discrete) whole life insurance

Consider a whole life insurance which pays benefit at the end of year of death (for life):

\[ b_{K+1} = 1, \quad v_{K+1} = v^{K+1}, \quad \text{and} \quad Z = v^{K+1}. \]

- **APV:** \( A_x = E[Z] = \sum_{k=0}^{\infty} v^{k+1} q_x = \sum_{k=0}^{\infty} v^{k+1} q_x \cdot q_{x+k} \)

- Applying rule of moments,

\[ \text{Var}[Z] = 2A_x - (A_x)^2, \]

where

\[ 2A_x = E[Z^2] = \sum_{k=0}^{\infty} e^{-2\delta(k+1)} k p_x \cdot q_{x+k}. \]
(Discrete) endowment life insurance

- The APV of a (discrete) endowment life insurance is the sum of the APV of a (discrete) term and a pure endowment:

\[ A_{x:n} = A_{x:n}^1 + A_{x:1/n} \]

- The policy pays a death benefit of $1 at the end of the year of death, if death is prior to the end of \( n \) years, and a benefit of $1 if the insured survives at least \( n \) years.

- In effect, we have \( b_{K+1} = 1 \) and \( v_{K+1} = \begin{cases} v^{K+1}, & K \leq n - 1 \\ v^n, & K \geq n \end{cases} \) so that the PV r.v. is \( Z = \begin{cases} v^{K+1}, & K \leq n - 1 \\ v^n, & K \geq n \end{cases} \).

- Here \( Z = v^{\min(K+1,n)} \) and one can also apply the rule of moments to evaluate the corresponding variance.
Recursive relationships

- The following will be derived/discussed in class:
  - whole life insurance: $A_x = vq_x + vp_x A_{x+1}$
  - term insurance: $A^{1}_{x:n} = vq_x + vp_x A^{1}_{x+1:n-1}$
  - endowment insurance: $A_{x:n} = vq_x + vp_x A_{x+1:n-1}$
Makeham parameters: 
\( A = 0.00022, \quad B = 2.7 \times 10^{-6}, \quad c = 1.124 \)

**Figure**: Actuarial Present Value of a discrete whole life insurance for various interest rate assumptions
Makeham parameters:
\( A = 0.00022, \ B = 2.7 \times 10^{-6}, \ c \) varying

\[ \begin{align*}
q_x & = 1.124 \\
c & = 1.130 \\
c & = 1.136 \\
c & = 1.142 \\
c & = 1.148
\end{align*} \]

\[ \begin{align*}
A_x & \approx \frac{1}{0.00022} \\
c & \approx 1.136 \\
B & \approx 2.7 \times 10^{-6}
\end{align*} \]

**Figure**: Actuarial Present Value of a discrete whole life insurance for various mortality rate assumptions with interest rate fixed at 5%
Illustrative example 3

For a whole life insurance of 1 on (41) with death benefit payable at the end of the year of death, let $Z$ be the present value random variable for this insurance.

You are given:

- $i = 0.05$;
- $p_{40} = 0.9972$;
- $A_{41} - A_{40} = 0.00822$; and
- $2A_{41} - 2A_{40} = 0.00433$.

Calculate $\text{Var}[Z]$. 
Other forms of insurance

- Deferred insurances
- Varying benefit insurances
- Very similar to the continuous cases
- You are expected to read and understand these other forms of insurances.
- It is also useful to understand the various (possible) recursion relations resulting from these various forms.
Illustration of varying benefits

For a special life insurance issued to (45), you are given:

- Death benefits are payable at the end of the year of death.
- The benefit amount is $100,000 in the first 10 years of death, decreasing to $50,000 after that until reaching age 65.
- An endowment benefit of $100,000 is paid if the insured reaches age 65.
- There are no benefits to be paid past the age of 65.
- Mortality follows the Illustrative Life Table at $i = 6\%$.

Calculate the actuarial present value (APV) for this insurance.
Illustrative example 4

For a whole life insurance issued to age 40, you are given:

- Death benefits are payable at the moment of death.
- The benefit amount is $1,000 in the first year of death, increasing by $500 each year thereafter for the next 3 years, and then becomes level at $5,000 thereafter.
- Mortality follows the Illustrative Life Table at $i = 6\%$.
- Deaths are uniformly distributed over each year of age.

Calculate the APV for this insurance.
Insurances payable \( m \)-thly

- Consider the case where we have just one-year term and the benefit is payable at the end of the \( m \)-th of the year of death.

- We thus have

\[
A^{(m)}_{1:x\backslash 1} = \sum_{r=0}^{m-1} \frac{v(r+1)/m}{\frac{r}{m} p_x \cdot \frac{1}{m} q_x + \frac{r}{m}}.
\]

- We can show that under the UDD assumption, this leads us to:

\[
A^{(m)}_{1:x\backslash 1} = \frac{i}{i(m)} A^{1}_{1:x\backslash 1}.
\]

- In general, we can generalize this to:

\[
A^{(m)}_{1:x\backslash n} = \frac{i}{i(m)} A^{1}_{x\backslash n}.
\]
Other types of insurances with \( m \)-thly payments

- For other types, we can also similarly derive the following (with the UDD assumption):
  - whole life insurance: \( A^{(m)}_x = \frac{i}{i(m)} A_x \)
  - deferred life insurance: \( n|A^{(m)}_x = \frac{i}{i(m)} n|A_x \)
  - endowment insurance: \( A^{(m)}_{x:n} = \frac{i}{i(m)} A^{1}_{x:n} + A^{1}_{x:n} \)
For some forms of insurances, we can get explicit relationships under the UDD assumption:

- whole life insurance: \( \bar{A}_x = \frac{i}{\delta} A_x \)

- term insurance: \( \bar{A}^{1}_{x:n} = \frac{i}{\delta} A^{1}_{x:n} \)

- increasing term insurance: \( (I\bar{A})^{1}_{x:n} = \frac{i}{\delta} (IA)^{1}_{x:n} \)
Illustrative example 5

For a three-year term insurance of 1000 on [50], you are given:

- Death benefits are payable at the end of the quarter of death.
- Mortality follows a select and ultimate life table with a two-year select period:

<table>
<thead>
<tr>
<th>[x]</th>
<th>(\ell_{[x]})</th>
<th>(\ell_{[x]+1})</th>
<th>(\ell_{x+2})</th>
<th>(x + 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>9706</td>
<td>9687</td>
<td>9661</td>
<td>52</td>
</tr>
<tr>
<td>51</td>
<td>9680</td>
<td>9660</td>
<td>9630</td>
<td>53</td>
</tr>
<tr>
<td>52</td>
<td>9653</td>
<td>9629</td>
<td>9596</td>
<td>54</td>
</tr>
</tbody>
</table>

- Deaths are uniformly distributed over each year of age.

- \(i = 5\%\)

Calculate the APV for this insurance.
Illustrative example 6

Each of 100 independent lives purchases a single premium 5-year deferred whole life insurance of 10 payable at the moment of death.

You are given:

- $\mu = 0.004$
- $\delta = 0.006$
- $F$ is the aggregate amount the insurer receives from the 100 lives.
- The 95th percentile of the standard Normal distribution is 1.645.

Using a Normal approximation, calculate $F$ such that the probability the insurer has sufficient funds to pay all claims is 0.95.
Illustrative example 7

Suppose interest rate \( i = 6\% \) and mortality is based on the following life table:

\[
\begin{array}{c|cccccccccc}
    x & 90 & 91 & 92 & 93 & 94 & 95 & 96 & 97 & 98 & 99 & 100 \\
    \ell_x & 800 & 740 & 680 & 620 & 560 & 500 & 440 & 380 & 320 & 100 & 0 \\
\end{array}
\]

Calculate the following:

(a) \( A_{94} \)

(b) \( A_{90:5}^{1} \)

(c) \( A_{92}^{(4)} \), assuming UDD between integral ages

(d) \( A_{95:3} \)
Illustrative example 8

A five-year term insurance policy is issued to (45) with benefit amount of $10,000 payable at the end of the year of death.

Mortality is based on the following select and ultimate life table:

<table>
<thead>
<tr>
<th>x</th>
<th>l_x</th>
<th>l_x+1</th>
<th>l_x+2</th>
<th>l_x+3</th>
<th>x + 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>5282</td>
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<td>4856</td>
<td>4600</td>
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</tr>
<tr>
<td>46</td>
<td>4753</td>
<td>4524</td>
<td>4322</td>
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<td>49</td>
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<td>47</td>
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<td>50</td>
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<tr>
<td>48</td>
<td>3816</td>
<td>3628</td>
<td>3480</td>
<td>3233</td>
<td>51</td>
</tr>
</tbody>
</table>

Calculate the APV for this insurance if $i = 5\%$. 
### Other terminologies and notations used

<table>
<thead>
<tr>
<th>Expression</th>
<th>Other terms/symbols used</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actuarial Present Value (APV)</td>
<td>Expected Present Value (EPV)</td>
</tr>
<tr>
<td></td>
<td>Net Single Premium (NSP)</td>
</tr>
<tr>
<td></td>
<td>single benefit premium</td>
</tr>
<tr>
<td>basis</td>
<td>assumptions</td>
</tr>
<tr>
<td>interest rate ((i))</td>
<td>interest per year effective</td>
</tr>
<tr>
<td></td>
<td>discount rate</td>
</tr>
<tr>
<td>benefit amount ((b))</td>
<td>sum insured ((S))</td>
</tr>
<tr>
<td></td>
<td>death benefit</td>
</tr>
<tr>
<td>Expected value of (Z)</td>
<td>(E(Z))</td>
</tr>
<tr>
<td>Variance of (Z)</td>
<td>(Var(Z)), (V[Z])</td>
</tr>
</tbody>
</table>