Life Tables and Selection

Lecture: Weeks 4-5
Chapter summary

- What is a life table?
  - also called a mortality table
  - tabulation of basic mortality functions
  - deriving probabilities/expectations from a life table

- Relationships to survival functions

- Assumptions for fractional (non-integral) ages

- Select and ultimate tables
  - national life tables
  - valuation or pricing tables

- Chapter 3, DHW
What is the life table?

- A tabular presentation of the mortality evolution of a cohort group of lives.
- Begin with $\ell_0$ number of lives (e.g. 100,000) - called the radix of the life table.
- (Expected) number of lives who are age $x$: $\ell_x = \ell_0 \cdot S_0(x) = \ell_0 \cdot x p_0$
- (Expected) number of deaths between ages $x$ and $x + 1$: $d_x = \ell_x - \ell_{x+1}$.
- (Expected) number of deaths between ages $x$ and $x + n$: $n d_x = \ell_x - \ell_{x+n}$.
- Conditional on survival to age $x$, the probability of dying within $n$ years is: $n q_x = n d_x / \ell_x = (\ell_x - \ell_{x+n}) / \ell_x$.
- Conditional on survival to age $x$, the probability of living to reach age $x + n$ is: $n p_x = 1 - n q_x = \ell_{x+n} / \ell_x$. 

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Example of a life table

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \ell_x )</th>
<th>( d_x )</th>
<th>( q_x )</th>
<th>( p_x )</th>
<th>( \hat{e}_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100,000</td>
<td>680</td>
<td>0.006799</td>
<td>0.993201</td>
<td>77.84</td>
</tr>
<tr>
<td>1</td>
<td>99,320</td>
<td>48</td>
<td>0.000483</td>
<td>0.999517</td>
<td>77.37</td>
</tr>
<tr>
<td>2</td>
<td>99,272</td>
<td>29</td>
<td>0.000297</td>
<td>0.999703</td>
<td>76.41</td>
</tr>
<tr>
<td>3</td>
<td>99,243</td>
<td>22</td>
<td>0.000224</td>
<td>0.999776</td>
<td>75.43</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>50</td>
<td>93,735</td>
<td>413</td>
<td>0.004404</td>
<td>0.995596</td>
<td>30.87</td>
</tr>
<tr>
<td>51</td>
<td>93,323</td>
<td>443</td>
<td>0.004750</td>
<td>0.995250</td>
<td>30.01</td>
</tr>
<tr>
<td>52</td>
<td>92,879</td>
<td>475</td>
<td>0.005113</td>
<td>0.994887</td>
<td>29.15</td>
</tr>
<tr>
<td>53</td>
<td>92,404</td>
<td>507</td>
<td>0.005488</td>
<td>0.994512</td>
<td>28.30</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>97</td>
<td>5,926</td>
<td>1,370</td>
<td>0.231201</td>
<td>0.768799</td>
<td>3.15</td>
</tr>
<tr>
<td>98</td>
<td>4,556</td>
<td>1,133</td>
<td>0.248600</td>
<td>0.751400</td>
<td>2.95</td>
</tr>
<tr>
<td>99</td>
<td>3,423</td>
<td>913</td>
<td>0.266786</td>
<td>0.733214</td>
<td>2.76</td>
</tr>
</tbody>
</table>

Source: U.S. Life Table for the total population, 2004, Center for Disease Control and Prevention (CDC)
Radix of the life table

- The radix of the life table does not have to start at age 0, e.g. start with age \( x_0 \), so that the table starts with radix \( \ell_{x_0} \).
- The limiting age of the table is usually denoted by \( \omega \), in which case the table ends at \( \omega - x_0 \).
- All the formulas still work, e.g. conditional on survival to age \( x \), the probability of surviving to reach age \( x + n \) is:
  \[
  nP_x = 1 - nq_x = \frac{\ell_{x+n}}{\ell_x}.
  \]
- Note that among \( \ell_x \) independent lives who have reached age \( x \), the number of survivors \( \mathcal{L}_n \) within \( n \) years is a Binomial random variable with parameters \( \ell_x \) and \( nP_x \) so that
  \[
  \mathbb{E}(\mathcal{L}_n) = \ell_x \cdot nP_x.
  \]
Revised example 3.1

Using Table 3.1, page 43 of DHW, calculate the following:

- the probability that (30) will survive another 5 years
- the probability that (39) will survive to reach age 40
- the probability that (30) will die within 10 years
- the probability that (30) dies between ages 36 and 38
**Illustrative example 1**

Complete the following life table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\ell_x$</th>
<th>$d_x$</th>
<th>$p_x$</th>
<th>$q_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>24,983</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>41</td>
<td>24,541</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>42</td>
<td>24,175</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>43</td>
<td>23,880</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>44</td>
<td>23,656</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>45</td>
<td>23,495</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

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Additional useful formulas

From a life table, the following formulas can also easily be verified (or use your intuition):

- \( \ell_x = \sum_{k=0}^{\infty} d_{x+k} \): the number of survivors at age \( x \) should be equal to the number of deaths in each year of age for all the following years.

- \( n d_x = \ell_x - \ell_{x+n} = \sum_{k=0}^{n-1} d_{x+k} \): the number of deaths within \( n \) years should be equal to the number of deaths in each year of age for the next \( n \) years.

- Finally, the probability that \( (x) \) survives the next \( n \) years but dies the following \( m \) years after that can be derived using:

\[
 n|m q_x = n p_x - n+m p_x = \frac{m d_{x+n}}{\ell_x} = \frac{\ell x+n - \ell x+n+m}{\ell x}.
\]
The force of mortality

- It is easy to show that the force of mortality can be expressed in terms of life table function as:

\[ \mu_x = - \frac{1}{\ell_x} \cdot \frac{d\ell_x}{dx} . \]

- Thus, in effect, we can also write

\[ \ell_x = \ell_0 \cdot \exp \left( - \int_0^x \mu_z \, dz \right) . \]

- With a simple change of variable, it is easy to see also that

\[ \mu_{x+t} = - \frac{1}{\ell_{x+t}} \cdot \frac{d\ell_{x+t}}{dt} = - \frac{1}{t\,p_x} \cdot \frac{d_t p_x}{dt} . \]

- It follows immediately that:

\[ \frac{d}{dt} t\,p_x = - t\,p_x \mu_{x+t} . \]
Curtate expectation of life

- Analogously, the expected value of \( K_x \), is called the curtate expectation of life defined by

\[
E[K_x] = e_x = \sum_{k=0}^{\infty} k k p_x q_{x+k}.
\]

- It can be shown (e.g. summation by parts) that

\[
e_x = \sum_{k=1}^{\infty} k p_x = \sum_{k=1}^{\infty} \frac{\ell_{x+k}}{\ell_x}.
\]

- The temporary curtate expectation of life defined by

\[
e_{x:n} = \sum_{k=1}^{n} k p_x = \sum_{k=1}^{n} \frac{\ell_{x+k}}{\ell_x},
\]

which gives the average number of completed years lived over the interval \((x, x + n]\) for a life \((x)\).
Illustrative example 2

Suppose you are given the following extract from a life table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\ell_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>94</td>
<td>16,208</td>
</tr>
<tr>
<td>95</td>
<td>10,902</td>
</tr>
<tr>
<td>96</td>
<td>7,212</td>
</tr>
<tr>
<td>97</td>
<td>4,637</td>
</tr>
<tr>
<td>98</td>
<td>2,893</td>
</tr>
<tr>
<td>99</td>
<td>1,747</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
</tr>
</tbody>
</table>

1. Calculate $e_{95}$.
2. Calculate the variance of $K_{95}$, the curtate future lifetime of (95).
3. Calculate $e_{95:3}$.
The life table typical human mortality curves

Figure: Source: Life Tables, 2007 from the Social Security Administration - male (blue), female (red)
Fractional age assumptions

When adopting a life table (which may contain only integer ages), some assumptions are needed about the distribution between the integers.

The two most common assumptions (or interpolations) used are (where $0 \leq t \leq 1$):

1. linear interpolation (also called UDD assumption):

$$\ell_{x+t} = (1 - t)\ell_x + t\ell_{x+1}$$

2. exponential interpolation (equivalent to constant force assumption):

$$\log \ell_{x+t} = (1 - t)\log \ell_x + t\log \ell_{x+1}$$
Some results on the fractional age assumptions

<table>
<thead>
<tr>
<th>Function</th>
<th>Linear (UDD)</th>
<th>Exponential (constant force)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t q_x )</td>
<td>( t \cdot q_x )</td>
<td>( 1 - (1 - q_x)^t )</td>
</tr>
<tr>
<td>( \mu_{x+t} )</td>
<td>( \frac{q_x}{1 - t \cdot q_x} )</td>
<td>( \mu = - \log p_x )</td>
</tr>
<tr>
<td>( t P_x \mu_{x+t} )</td>
<td>( q_x )</td>
<td>( \mu e^{-\mu t} )</td>
</tr>
</tbody>
</table>

Here we have \( 0 \leq t \leq 1 \).
Illustrative example 3

You are given the following extract from a life table:

\[
\begin{array}{cc}
 x & \ell_x \\
 55 & 85,916 \\
 56 & 84,772 \\
 57 & 83,507 \\
 58 & 82,114 \\
\end{array}
\]

Estimate \( 1.4p_{55} \) and \( 0.5|1.6q_{55} \) under each of the following assumptions for non-integral ages:

(a) UDD; and

(b) constant force.

Interpret these probabilities.
Fractional part of the year lived

- Denote by $R_x$ the fractional part of a year lived in the year of death. Then we have

$$T_x = K_x + R_x$$

where $T_x$ is the time-until-death and $K_x$ is the curtate future lifetime of $(x)$.

- We can describe the joint probability distribution of $(K_x, R_x)$ as

$$\Pr[(K_x = k) \cap (R_x \leq s)] = \Pr[k < T_x \leq k + s] = k p_x \cdot s q_{x+k},$$

for $k = 0, 1, \ldots$ and for $0 < s < 1$.

- The UDD assumption is equivalent to the assumption that the fractional part $R_x$ occurs uniformly during the year, i.e. $R_x \sim U(0, 1)$.

- It can be demonstrated that $K_x$ and $R_x$ are independent in this case.
Select and ultimate tables

- Group of lives underwritten for insurance coverage usually has different mortality than the general population (some test required before insurance is offered).

- Mortality then becomes a function of age \([x]\) at selection (e.g. policy issue, onset of disability) and duration \(t\) since selection.

- For select tables, notation such as \(tq[x]\), \(tp[x]\), and \(\ell[x] + t\), are then used.

- However, impact of selection diminishes after some time - the select period (denoted by \(r\)).

- In effect, we have

\[
q[x] + j = q[x] + j, \text{ for } j \geq r.
\]
Example of a select and ultimate table

<table>
<thead>
<tr>
<th>$x$</th>
<th>$1000q_x$</th>
<th>$1000q_{x+1}$</th>
<th>$1000q_{x+2}$</th>
<th>$\ell_x$</th>
<th>$\ell_{x+1}$</th>
<th>$\ell_{x+2}$</th>
<th>$x+2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.222</td>
<td>0.330</td>
<td>0.422</td>
<td>9,907</td>
<td>9,905</td>
<td>9,901</td>
<td>32</td>
</tr>
<tr>
<td>31</td>
<td>0.234</td>
<td>0.352</td>
<td>0.459</td>
<td>9,903</td>
<td>9,901</td>
<td>9,897</td>
<td>33</td>
</tr>
<tr>
<td>32</td>
<td>0.250</td>
<td>0.377</td>
<td>0.500</td>
<td>9,899</td>
<td>9,896</td>
<td>9,893</td>
<td>34</td>
</tr>
<tr>
<td>33</td>
<td>0.269</td>
<td>0.407</td>
<td>0.545</td>
<td>9,894</td>
<td>9,892</td>
<td>9,888</td>
<td>35</td>
</tr>
<tr>
<td>34</td>
<td>0.291</td>
<td>0.441</td>
<td>0.596</td>
<td>9,889</td>
<td>9,887</td>
<td>9,882</td>
<td>36</td>
</tr>
</tbody>
</table>

- From this table, try to compute probabilities such as:
  
  (a) $2p_{30}$;
  
  (b) $5p_{30}$;
  
  (c) $1|q_{31}$; and
  
  (d) $3q_{31}+1$. 

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Illustrative example 4

A select and ultimate table with a three-year select period begins at selection age $x$.

You are given the following information:

- $\ell_{x+6} = 90,000$
- $q[x] = \frac{1}{6}$
- $5p[x+1] = \frac{4}{5}$
- $3p[x]+1 = \frac{9}{10} \cdot 3p[x+1]$.

Evaluate $\ell[x]$. 


Illustrative example 5

You are given the following extract from a select and ultimate life table:

<table>
<thead>
<tr>
<th></th>
<th>( \ell_x )</th>
<th>( \ell_{x+1} )</th>
<th>( \ell_{x+2} )</th>
<th>( x + 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>29,616</td>
<td>29,418</td>
<td>29,132</td>
<td>62</td>
</tr>
<tr>
<td>61</td>
<td>29,131</td>
<td>28,920</td>
<td>28,615</td>
<td>63</td>
</tr>
<tr>
<td>62</td>
<td>28,601</td>
<td>28,378</td>
<td>28,053</td>
<td>64</td>
</tr>
</tbody>
</table>

Calculate \( 1000 \cdot q_{[60]+0.8} \), assuming a constant force of mortality at fractional ages.
Illustrative example 6

You are given the following extract from a select and ultimate life table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\ell_x$</th>
<th>$\ell_{x+1}$</th>
<th>$\ell_{x+2}$</th>
<th>$x+2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>80,625</td>
<td>79,954</td>
<td>78,839</td>
<td>67</td>
</tr>
<tr>
<td>66</td>
<td>79,137</td>
<td>78,402</td>
<td>77,252</td>
<td>68</td>
</tr>
<tr>
<td>67</td>
<td>77.575</td>
<td>76,770</td>
<td>75,578</td>
<td>69</td>
</tr>
</tbody>
</table>

Approximate $\hat{\delta}_{65:2}$ using the trapezium (trapezoidal) rule with $h = 0.5$ and assuming UDD for fractional ages.
Illustrative example 7

The mortality pattern of a life \((x)\) is based on a select and ultimate survival model where the ultimate part follows De Moivre’s law with \(\omega = 80\).

You are given:

\[
q[x] + t = \begin{cases} 
\frac{t+1}{t+2}q[x] + t, & t = 0, 1, 2 \\
q[x] + t, & t = 3, 4, \ldots 
\end{cases}
\]

Calculate the probability that an individual, insured (or selected) one year ago at age 35, will die between age 38 and 40.
Suppose you are given:

- $p_{50} = 0.98$
- $p_{51} = 0.96$
- $e_{51.5} = 22.4$
- The force of mortality is constant between ages 50 and 51.
- Deaths are uniformly distributed between ages 51 and 52.

Calculate $e_{50.5}$. 
Illustrative example 9 - modified SOA MLC Spring 2012

In a 2-year select and ultimate mortality table, you are given:

- \( q_{[x]+1} = 0.96 \ q_{x+1} \)
- \( \ell_{65} = 82,358 \)
- \( \ell_{66} = 81,284 \)

Calculate \( \ell_{[64]+1} \).
Mortality projection factors

Read Section 3.11
Only other symbol used in the MLC exam

<table>
<thead>
<tr>
<th>Expression</th>
<th>SOA will adopt</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of lives</td>
<td>$l_x$</td>
</tr>
</tbody>
</table>