Optimal Capital Allocation Principles
joint work with J. Dhaene, A. Tsanakas and S. Vanduffel

Emiliano A. Valdez, University of Connecticut

Katholieke Universiteit - Leuven, Belgium
9 June 2009
The allocation of capital

Capital allocation is the term usually referring to the subdivision of a company’s aggregate capital across its various constituents:

- lines of business
- its subsidiaries
- product types within lines of business
- territories, e.g. distribution channels
- types of risks: e.g. market, credit, pricing/underwriting, operational

A very important component of Enterprise Risk Management:

- identifying, measuring, pricing and controlling risks
Figure: the allocation by lines of business

Company Capital \( K \)

- Line of Business 1
- Line of Business 2
- Line of Business \( n \)
The literature

There are countless number of ways to allocate aggregate capital. Good overview of methods:

- Cummins (2000); Venter (2004)

Some methods based on decision making tools:

- Cummins (2000)- RAROC, EVA
- Lemaire (1984); Denault (2001) - game theory
- Tasche (2004) - marginal costs
- Kim and Hardy (2008) - solvency exchange option with limited liability
Some methods based on risk measures/distributions:

- Panjer (2001) - TVaR, multivariate normal
- Landsman and Valdez (2003) - TVaR, multivariate elliptical
- Dhaene, et al. (2008) - TVaR, lognormal
- Valdez and Chernih (2003) - covariance-based allocation, multivariate elliptical
- Tsanakas (2004, 2008) - distortion risk measures, convex risk measures
- Furman and Zitikis (2008) - weighted risk capital allocations

Methods also based on optimization principle:

- Dhaene, Goovaerts and Kaas (2003); Laeven and Goovaerts (2004); Zaks, Frostig and Levikson (2006)
The allocation problem

- Consider a portfolio of \( n \) individual losses \( X_1, \ldots, X_n \) during some well-defined reference period.

- Assume these random losses have a dependency structure characterized by the joint distribution of the random vector \((X_1, \ldots, X_n)\).

- The aggregate loss is the sum \( S = \sum_{i=1}^{n} X_i \).

- Assume company holds aggregate level of capital \( K \) which may be determined from a risk measure \( \rho \) such that \( K = \rho(S) \in \mathbb{R} \).

- Here the capital (economic) is the smallest amount the company must set aside to withstand aggregated losses at an acceptable level.
• The company now wishes to allocate $K$ across its various business units.

  • determine non-negative real numbers $K_1, \ldots, K_n$ satisfying:

  \[ \sum_{i=1}^{n} K_i = K. \]

• This requirement is referred to as “the full allocation” requirement.

• We will see that this requirement is a constraint in our optimization problem.
Our contribution to the literature

- We re-formulate the problem as minimum distance problem in the sense that the weighted sum of measure for the deviations of the business unit’s losses from their respective capitals be minimized:
  - essentially distances between $K_j$ and $X_j$
- Takes then into account some important decision making allocation criteria such as:
  - the purpose of the allocation allowing the risk manager to meet specific target objectives
  - the manner in which the various segments interact, e.g. legal structure
- Solution to minimizing distance formula leads to several existing allocation methods. New allocation formulas also emerge.
Risk measures

A risk measure is a mapping $\rho$ from a set $\Gamma$ of real-valued r.v.'s defined on $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}$:

$$\rho : \Gamma \rightarrow \mathbb{R} : X \in \Gamma \rightarrow \rho[X].$$

Let $X, X_1, X_2 \in \Gamma$. Some well known properties that risk measures may or may not satisfy:

- **Law invariance**: If $\mathbb{P}[X_1 \leq x] = \mathbb{P}[X_2 \leq x]$ for all $x \in \mathbb{R}$, $\rho[X_1] = \rho[X_2]$.

- **Monotonicity**: $X_1 \leq X_2$ implies $\rho[X_1] \leq \rho[X_2]$.

- **Positive homogeneity**: For any $a > 0$, $\rho[aX] = a\rho[X]$.

- **Translation invariance**: For $b \in \mathbb{R}$, $\rho[X + b] = \rho[X] + b$.

- **Subadditivity**: $\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2]$. 
For $p \in (0, 1)$, we denote the Value-at-Risk (VaR) or quantile of $X$ by $F_X^{-1}(p)$ defined by:

$$F_X^{-1}(p) = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}.$$ 

We define the inverse distribution function $F_X^{-1+}(p)$ of $X$ as

$$F_X^{-1+}(p) = \sup \{x \in \mathbb{R} \mid F_X(x) \leq p\}.$$

The $\alpha$–mixed inverse distribution function $F_X^{-1}(\alpha)$ of $X$ is:

$$F_X^{-1}(\alpha)(p) = \alpha F_X^{-1}(p) + (1 - \alpha) F_X^{-1+}(p).$$

It follows for any $X$ and for all $x$ with $0 < F_X(x) < 1$, there exists an $\alpha_x \in [0, 1]$ such that $F_X^{-1}(\alpha_x)(F_X(x)) = x$. 

Some important concepts

Conditional Tail Expectation (CTE): (sometimes called TailVaR)

$$CTE_p [X] = \mathbb{E} [X \mid X > F_X^{-1}(p)] , \quad p \in (0, 1).$$

In general, not subadditivite, but so for continuous random variables.

Comonotonic sum: \( S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U) \) where \( U \) is uniform on \((0, 1)\).

The Fréchet bounds:

$$L_F(u_1, \ldots, u_n) \leq C(u_1, \ldots, u_n) \leq U_F(u_1, \ldots, u_n),$$

where

Fréchet lower bound: \( L_F = \max(\sum_{i=1}^{n} u_i - (n - 1), 0) \), and

Fréchet upper bound: \( U_F = \min(u_1, \ldots, u_n) \).
Some known allocation formulas

Many well-known allocation formulas fall into a class of proportional allocations.

Members of this class are obtained by first choosing a risk measure $\rho$ and then attributing the capital $K_i = \gamma \rho [X_i]$ to each business unit $i$, $i = 1, \ldots, n$.

The factor $\gamma$ is chosen such that the full allocation requirement is satisfied.

This gives rise to the proportional allocation principle:

$$K_i = \frac{K}{\sum_{j=1}^{n} \rho[X_j]} \rho[X_i], \quad i = 1, \ldots, n.$$
## Some known allocation formulas

<table>
<thead>
<tr>
<th>Allocation method</th>
<th>$\rho[X_i]$</th>
<th>$K_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Haircut allocation (no known reference)</td>
<td>$F_{X_i}^{-1}(p)$</td>
<td>$\frac{K}{\sum_{j=1}^{n} F_{X_j}^{-1}(p)} F_{X_i}^{-1}(p)$</td>
</tr>
<tr>
<td>Quantile allocation</td>
<td>$F_{X_i}^{-1(\alpha)} (F_{S_c}(K))$</td>
<td>$F_{X_i}^{-1(\alpha)} (F_{S_c}(K))$</td>
</tr>
<tr>
<td>Dhaene et al. (2002)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Covariance allocation</td>
<td>$\text{Cov}[X_i, S]$</td>
<td>$\frac{K}{\text{Var}[S]} \text{Cov}[X_i, S]$</td>
</tr>
<tr>
<td>Overbeck (2000)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CTE allocation</td>
<td>$\mathbb{E}[X_i \mid S &gt; F_s^{-1}(p)]$</td>
<td>$\frac{K}{\text{CTE}_p[S]} \mathbb{E}[X_i \mid S &gt; F_s^{-1}(p)]$</td>
</tr>
<tr>
<td>Acerbi and Tasche (2002), Dhaene et al. (2006)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The optimal capital allocation problem

We reformulate the allocation problem in terms of optimization:

Given the aggregate capital \( K > 0 \), we determine the allocated capitals \( K_i, \ i = 1, \ldots, n \), from the following optimization problem:

\[
\min_{K_1, \ldots, K_n} \sum_{j=1}^{n} v_j \mathbb{E} \left[ \zeta_j D \left( \frac{X_j - K_j}{v_j} \right) \right]
\]

such that the full allocation is met:

\[
\sum_{j=1}^{n} K_j = K,
\]

and where the \( v_j \)'s are non-negative real numbers such that \( \sum_{j=1}^{n} v_j = 1 \), the \( \zeta_j \) are non-negative random variables such that \( \mathbb{E}[\zeta_j] = 1 \) and \( D \) is a non-negative function.
The components of the optimization

Elaborating on the various elements of the optimization problem:

- **Distance measure:** the function $D(\cdot)$ gives the deviations of the outcomes of the losses $X_j$ from their allocated capitals $K_j$.
  
  - squared-error or quadratic: $D(x) = x^2$
  
  - absolute deviation: $D(x) = |x|$

- **Weights:** the random variable $\zeta_j$ provides a re-weighting of the different possible outcomes of these deviations.

- **Exposure:** the non-negative real number $\nu_j$ measures exposure of each business unit according to for example, revenue, premiums, etc.
The case of the quadratic optimization

In the special case where

\[ D(x) = x^2 \]

so that the optimization is expressed as

\[
\min_{K_1, \ldots, K_n} \sum_{j=1}^{n} \mathbb{E} \left[ \zeta_j \frac{(X_j - K_j)^2}{v_j} \right].
\]

This optimal allocation problem has the following unique solution:

\[
K_i = \mathbb{E}[\zeta_i X_i] + v_i \left( K - \sum_{j=1}^{n} \mathbb{E}[\zeta_j X_j] \right), \quad i = 1, \ldots, n.
\]
### Business unit driven allocations

<table>
<thead>
<tr>
<th>Risk measure</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Standard deviation principle</strong></td>
<td>$h_i(X_i) = 1 + a \frac{X_i - \mathbb{E}[X_i]}{\sigma_{X_i}}$, $a \geq 0$</td>
</tr>
<tr>
<td>Buhlmann (1970)</td>
<td>$\mathbb{E}[X_i] + a\sigma_{X_i}$</td>
</tr>
<tr>
<td><strong>Conditional tail expectation</strong></td>
<td>$CTE_p[X_i] = \frac{1}{1 - p} \mathbb{I}(X_i &gt; F_{X_i}^{-1}(p)), p \in (0, 1)$</td>
</tr>
<tr>
<td>Overbeck (2000)</td>
<td>$CTE_p[X_i] = \mathbb{E}[X_i] + a\sigma_{X_i}$</td>
</tr>
<tr>
<td><strong>Distortion risk measure</strong></td>
<td>$g'(\overline{F}_{X_i}(X_i))$, $g : [0, 1] \to [0, 1]$, $g' &gt; 0$, $g'' &lt; 0$</td>
</tr>
<tr>
<td>Wang (1996), Acerbi (2002)</td>
<td>$\mathbb{E}[X_ig'(\overline{F}_{X_i}(X_i))]$</td>
</tr>
<tr>
<td><strong>Exponential principle</strong></td>
<td>$\int_0^1 \frac{e^{\gamma aX_i}}{\mathbb{E}[e^{\gamma aX_i}]} d\gamma, a &gt; 0$</td>
</tr>
<tr>
<td>Gerber (1974)</td>
<td>$\frac{1}{a} \ln \mathbb{E}[e^{aX_i}]$</td>
</tr>
<tr>
<td><strong>Esscher principle</strong></td>
<td>$\frac{e^{aX_i}}{\mathbb{E}[e^{aX_i}], a &gt; 0}$</td>
</tr>
<tr>
<td>Gerber (1981)</td>
<td>$\frac{\mathbb{E}[X_ie^{aX_i}]}{\mathbb{E}[e^{aX_i}]}$</td>
</tr>
</tbody>
</table>
## Aggregate portfolio driven allocations

<table>
<thead>
<tr>
<th>Reference</th>
<th>Expression</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overbeck (2000)</td>
<td>$\zeta_i = h(S)$, $E[X_i h(S)]$</td>
<td>$E[X_i] + a \frac{\text{Cov}[X_i, S]}{\sigma_S}$</td>
</tr>
<tr>
<td>Overbeck (2000)</td>
<td>$1 + a \frac{S - E[S]}{\sigma_S}$, $a \geq 0$</td>
<td>$E[X_i] + a \frac{\text{Cov}[X_i, S]}{\sigma_S}$</td>
</tr>
<tr>
<td>Overbeck (2000)</td>
<td>$\frac{1}{1 - p} \mathbb{I}(S &gt; F_S^{-1}(p))$, $p \in (0, 1)$</td>
<td>$E[X_i</td>
</tr>
<tr>
<td>Tsanakas (2004)</td>
<td>$g'(F_S(S))$, $g : [0, 1] \mapsto [0, 1]$, $g' &gt; 0$, $g'' &lt; 0$</td>
<td>$E [X_i g'(F_S(S))]$</td>
</tr>
<tr>
<td>Tsanakas (2008)</td>
<td>$\int_0^1 \frac{e^{\gamma a S}}{E[e^{\gamma a S}]} d\gamma$, $a &gt; 0$</td>
<td>$E \left[ X_i \int_0^1 \frac{e^{\gamma a S}}{E[e^{\gamma a S}]} d\gamma \right]$</td>
</tr>
<tr>
<td>Wang2007</td>
<td>$\frac{e^{a S}}{E[e^{a S}]}$, $a &gt; 0$</td>
<td>$E[X_i e^{a S}] / E[e^{a S}]$</td>
</tr>
</tbody>
</table>
Market driven allocations

Let \( \zeta_M \) be such that market-consistent values of the aggregate portfolio loss \( S \) and the business unit losses \( X_i \) are given by
\[
\pi[S] = \mathbb{E}[\zeta_M S] \quad \text{and} \quad \pi[X_i] = \mathbb{E}[\zeta_M X_i].
\]
To determine an optimal allocation over the different business units, we let \( \zeta_i = \zeta_M, \quad i = 1, \ldots, n \), allowing the market to determine which states-of-the-world are to be regarded adverse. This yields:
\[
K_i = \pi[X_i] + v_i (K - \pi[S]).
\]
Using market-consistent prices as volume measures \( v_i = \pi[X_i]/\pi[S] \), we find
\[
K_i = \frac{K}{\pi[S]} \pi[X_i], \quad i = 1, \ldots, n.
\]
Rearranging these expressions leads to
\[
\frac{K_i - \pi[X_i]}{\pi[X_i]} = \frac{K - \pi[S]}{\pi[S]}, \quad i = 1, \ldots, n.
\]
Additional items considered in the paper

- Allocation according to the default option.
  - $\zeta_i$ is suitably chosen to account for shareholders having limited liability - not obligated to pay excess $(S - K)$ in case of default.

- We also considered other optimization criterion:
  - absolute value deviation: $D(x) = |x|$
  - combined quadratic/shortfall: $D(x) = ((x)_+)^2$
  - shortfall: $D(x) = (x)_+$

- Shortfall is applicable in cases where insurance market guarantees payments out of a pooled fund contributed by all companies, e.g. Lloyd’s.

- Such allocation can be posed as an optimization problem leading to formulas that have been considered by Lloyd’s. [Note: views here are the authors’ own and do not necessarily reflect those of Lloyd’s.]
Concluding remarks

- We re-examine existing allocation formulas that are in use in practice and existing in the literature. We re-express the allocation issue as an optimization problem.

- No single allocation formula may serve multiple purposes, but by expressing the problem as an optimization problem it can serve us more insights.

- Each of the components in the optimization can serve various purposes.

- This allocation methodology can lead to a wide variety of other allocation formulas.
Thank you.