MTH 235 Ordinary Differential Equations

Review for Final Exam

1. Find the general solution to the following second order ODE

$$y'' - 8y' + 16y = \frac{-3e^{4t}}{t^2}.$$

Answer: $y(t) = c_1 e^{4t} + c_2 t e^{4t} + 3\ln(t)e^{4t}$

2. Find the solution to the following second order IVP, using Laplace Transform

$$y'' + 6y' + 10y = 0, \quad y(0) = -5, \quad y'(0) = -4.$$

Answer: $Y(s) = \frac{-5(s+3) - 19}{(s+3)^2 + 1}, \quad y(t) = -5e^{-3t}\cos(t) - 19e^{-3t}\sin(t)$

3. Find the solution to the following IVP

$$y' = 2y + 2te^{2t}, \qquad y(0) = 5.$$

Answer: $y(t) = 5e^{2t} + t^2 e^{2t}$

4. Find the general solution to the following ODE

$$y' - 8y^2\cos(t) - 7y^2\sin(4t) = 0.$$

Answer: $y(t) = \left(-8\sin(t) + \frac{7}{4}\cos(4t) + c\right)^{-1}$

5. Find the solution to the following second order IVP, using Laplace Transform

$$y'' - 8y' + 16y = 5\delta(t-3), \quad y(0) = 0, \quad y'(0) = 0.$$

Answer: $y(t) = 5u(t-3)\cdot(t-3)e^{4(t-3)}$

6. Find the solution to the following second order IVP

$$y'' - 5y' + 4y = 0, \qquad y(0) = -5, \quad y'(0) = 3.$$

Answer: $y(t) = \frac{8}{3}e^{4t} - \frac{23}{3}e^{t}$

7. Find the solution to the following second order IVP

$$y'' - 8y' + 32y = 0, \qquad y(0) = -2, \quad y'(0) = -4.$$

Answer: $y(t) = -2e^{4t}\cos(4t) + e^{4t}\sin(4t)$

8. Find the solution to the following second order IVP

$$y'' - 8y' + 15y = 4e^{t}, \qquad y(0) = 5, \quad y'(0) = 1.$$

Answer: $y(t) = -\frac{13}{2}e^{5t} + 11e^{3t} + \frac{4}{8}e^{t}$

9. Find the general solution to the following second order ODE

$$y'' - 6y' + 8y = 3e^{2t}.$$

Answer: $y(t) = c_1 e^{4t} + c_2 e^{2t} - \frac{3}{2} t e^{2t}$

10. Find the solution to the following initial value problem

$$y' + 8y^3 \cos(7t) = 0, \qquad y(0) = 2.$$

Answer: $y(t) = \left(\frac{16}{7}\sin(7t) + \frac{1}{4}\right)^{-1/2}$

11. Find the general solution to the following second order ODE

$$y'' - 10y' + 24y = -3\sin(2t).$$

Answer: $y(t) = c_1 e^{4t} + c_2 e^{6t} - \frac{3}{40}(\cos(2t) + \sin(2t)))$

12. Consider the following second order IVP with an arbitrary force term, $g(t)$

$$y'' - 4y' + 20y = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Let $G(s) = \mathcal{L}[g]$ and $Y(s) = \mathcal{L}[y]$. Find $H(s)$, such that $Y(s) = H(s)G(s)$ and $h(t)$ such that $y(t) = h \star g(t)$.

Answer: $H(s) = \frac{1}{(s-2)^2 + 4^2}$, $h(t) = e^{2t}\frac{\sin(4t)}{4}$

13. Find the solution to the following IVP

$$ty' = 2y - 3t^3 \cos(4t), \qquad y(\pi/8) = 0.$$

Answer: $y(t) = \frac{3}{4}t^2 - \frac{3}{4}t^3 \sin(4t)$

14. Find the Laplace Transform of the following function.

$$f(t) = \begin{cases} 0, & t < 3 \\ t^2 - 6t + 7, & t \geq 3. \end{cases}$$

Answer: $\mathcal{L}[f](s) = e^{-3s}\left(\frac{2}{s^3} - \frac{2}{s}\right)$

15. Find the solution to the following IVP

$$y' = \tan(t)y - 5t, \quad t \in [0, \frac{\pi}{2}), \qquad y(0) = 3.$$

Answer: $y(t) = \frac{8}{\cos(t)} - 5t\tan(t) - 5$

16. Find the solution to the following second order IVP, using Laplace Transform

$$y'' - 7y' + 12y = 5u(t-3)e^{-3(t-3)}, \quad y(0) = 0, \quad y'(0) = 0.$$

Answer: $Y(s) = \frac{5e^{-3s}}{(s+3)(s^2 - 7s + 12)}$, $y(t) = 5u(t-3)\left(-\frac{1}{6}e^{3(t-3)} + \frac{1}{42}e^{-3(t-3)} + \frac{1}{7}e^{4(t-3)}\right)$

17. Find the general solution to the following second order ODE

$$y'' - 8y' + 16y = 0.$$

Answer: $y(t) = c_1 e^{4t} + c_2 t e^{4t}$

18. Consider the matrix

$$A = \begin{bmatrix} -2 & 2 \\ -4 & 4 \end{bmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Answer: $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

19. Find the solution to the system $\mathbf{x}' = A\mathbf{x}$ of ODEs with the given initial condition and where A is the given below.

(a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} -2 \\ -5 \end{bmatrix},$

Answer: $\mathbf{x}(t) = 7 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{3t} - 9 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

(b) $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -4 \end{bmatrix},$

Answer: $\mathbf{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} e^{2t} + 4 \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix} e^{2t}$

(c) $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} -5 \\ -3 \end{bmatrix},$

Answer: $\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} - 8 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{3t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} \right)$

20. Consider the following second order IVP $y'' + y' + 5y = -4\cos(5t), \quad y(0) = -3, \quad y'(0) = 2.$ Write it as a first order system of the form $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} y \\ y' \end{bmatrix}$.

Answer: $A = \begin{bmatrix} 0 & 1 \\ -5 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -4\cos(5t) \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

21. Consider the following system of ODEs.

$$\begin{aligned} u' &= -u + 2v \\ v' &= 2u - v, \end{aligned}$$

with initial conditions $u(0) = 3$, $v(0) = 2$. Find a second order IVP for the function u.

Answer: $u'' + 2u' - 3u = 0, u(0) = 3, u'(0) = 1.$

22. Consider the following nonlinear system. Find all equilibrium points, find the matrix of the linearization around this equilibrium, and determine their type and stability.

$$\begin{aligned} x' &= 4y - y^3 \\ y' &= -9x - y^2. \end{aligned}$$

Answer: $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $A_0 = \begin{bmatrix} 0 & 4 \\ -9 & 0 \end{bmatrix}$, $\mathbf{x}_0$ is a center; $\mathbf{x}_1 = \begin{bmatrix} -\frac{4}{9} \\ 2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & -8 \\ -9 & -4 \end{bmatrix}$, $\mathbf{x}_1$ is a saddle; $\mathbf{x}_2 = \begin{bmatrix} -\frac{4}{9} \\ -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & -8 \\ -9 & 4 \end{bmatrix}$, $\mathbf{x}_2$ is a saddle;

23. Find the eigenvalues λ_n and corresponding (nonzero) eigenfunctions y_n, which solve

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(4) = 0.$$

Answer: $\lambda_n = \left(\frac{(2n-1)\pi}{8}\right)^2, \quad y_n(x) = \sin\left(\frac{(2n-1)\pi}{8}x\right).$

24. Find the Fourier series of the following function

$$f(x) = 2x + 5, \quad x \in [-3, 3].$$

Answer: $5 + \sum_{n=1}^{\infty} \left(-12\frac{\cos(n\pi)}{n\pi}\right) \sin\left(\frac{n\pi x}{3}\right).$

25. Let u be the solution to the following initial boundary value problem for the Heat Equation

$$\partial_t u(t,x) = 3\partial_x^2 u(t,x), \quad t > 0, \quad x \in (0,3),$$

with an initial condition $u(0,x) = f(x)$ and with boundary conditions $\partial_x u(t,0) = 0$ and $\partial_x u(t,3) = 0$. Let the functions $v_n(t)$ and $w_n(x)$ in the expansion $u(t,x) = \sum_{n=1}^{\infty} c_n v_n(t) w_n(x)$.

Answer: $v_n(t) = e^{-3\left(\frac{n\pi}{3}\right)^2 t}, \quad w_n(x) = \cos\left(\frac{n\pi x}{3}\right)$