4.4. Generalized Sources

Section Objective(s):
• The Dirac’s Delta.
• Main Properties.
• Applications.
• The Impulse Response Function.

4.4.1. The Dirac Delta.

Definition 4.4.1. The Dirac delta generalized function is the limit

\[ \delta_n(t) = \lim_{n \to \infty} \delta_n(t), \]

for every fixed \( t \in \mathbb{R} \) of the sequence functions \( \{\delta_n\}_{n=1}^{\infty} \).

Remark: The sequence of bump functions introduced above can be rewritten as follows,

\[ \delta_n(t) = \begin{cases} 
0, & t < 0 \\
0, & 0 \leq t < \frac{1}{n} \\
\infty, & t \geq \frac{1}{n}.
\end{cases} \]

We then obtain the equivalent expression,

\[ \delta(t) = \begin{cases} 
0, & t \neq 0, \\
\infty, & t = 0.
\end{cases} \]

Remark: There are infinitely many sequences \( \{\delta_n\} \) of functions with the Dirac delta as their limit as \( n \to \infty \).

Remarks:
(a) The Dirac delta is \underline{on the domain} \underline{domain}.
(b) The Dirac delta is \underline{on \underline{domain}}.

Theorem. Every function in the sequence \( \{\delta_n\} \) above satisfies

\[ \int_{c}^{c+1} \delta_n(t) \, dt = 1. \]
4.4.2. Main Properties.

Remark: We use ________ to define operations on Dirac’s deltas.

**Definition 4.4.2.** We introduce the following operations on the Dirac delta:

\[ f(t) \delta(t - c) + g(t) \delta(t - c) = \] 

\[ \int_a^b \delta(t - c) \, dt = \] 

\[ L[\delta(t - c)] = \] 

**Theorem 4.4.3.** For every \( c \in \mathbb{R} \) and \( \epsilon > 0 \) holds,

\[ \text{□} \]

Proof of Theorem 4.4.3:
Theorem 4.4.4. If $f$ is continuous on $(a,b)$ and $c \in (a,b)$, then

\[
\int_{a}^{b} f(t) \delta(t-c) \, dt = f(c).
\]

Proof of Theorem 4.4.4:
Theorem 4.4.5. For all $s \in \mathbb{R}$ holds

$$\mathcal{L}[\delta(t - c)] = \begin{cases} e^{-cs} & \text{for } c \geq 0, \\ 0 & \text{for } c < 0. \end{cases}$$

Proof of Theorem 4.4.5:
4.4.3. Applications of the Dirac Delta.

Remarks:

(a) Dirac’s delta generalized function is useful to describe

……………………………………………………………………………………………………………………………

(b) An impulsive force transfers a……………………………………………………………………………………

……………………………………………………………………………………………………………………………

(c) For example, a pendulum at rest that is hit by a hammer.
**Example 1:** Use Newton’s equation of motion and Dirac’s delta to describe the change of momentum when a particle is hit by a hammer.

**Solution:**
4.4.4. The Impulse Response Function.

**Definition 4.4.6.** The *impulse response function* at the point $c \geq 0$ of the linear operator

\[
L(y) = y'' + a_1 y' + a_0 y,
\]

with $a_1$, $a_0$ constants, is the solution $y_\delta$ of

\[
L(y_\delta) = \delta(t - c), \quad y_\delta(0) = 0, \quad y'_\delta(0) = 0.
\]

**Theorem 4.4.7.** The function $y_\delta$ is the impulse response function at $c \geq 0$ of the constant coefficients operator $L(y) = y'' + a_1 y' + a_0 y$ iff holds

\[
L[y] = e^{-cs} p(s)
\]

where the characteristic polynomial $p(s)$ of $L$.

**Remark:** The impulse response function $y_\delta$ at $c = 0$ satisfies

\[
L[y] = e^{-cs} p(s)
\]

**Proof of Theorem 4.4.7:**
**Example 2:** Compare the solutions to the following two IVPs, by showing that their Laplace Transforms are the same.

\[ y'' + a_1 y' + a_0 y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0. \]

and

\[ y'' + a_1 y' + a_0 y = 0, \quad y(0) = 0, \quad y'(0) = 1. \]

Provide physics-based explanation of why these solutions coincide.

**Example 3:** Find the solution \( y \) to the initial value problem

\[ y'' - y = \delta(t - 3), \quad y(0) = 0, \quad y'(0) = 0. \]

**Solution:**