**Definition.** A metric on a space $X$ is given by a function $d : X \times X \to \mathbb{R}$, which satisfies all of the following conditions:

1. $\forall x, y \in X, d(x, y) \geq 0,$
2. $d(x, y) = 0$ if and only if $x = y$,
3. $\forall x, y \in X, d(x, y) = d(y, x),$
4. $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z).$

**Problem 1.** Show that any norm defines a metric by $d(x, y) = ||x - y||$.

The discrete metric $\rho$ on a space $X$ is defined by

$$\rho(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

**Problem 2.** Prove that $\rho$ is indeed a metric (i.e. satisfies conditions 1-4 above).

**Problem 3.** Does any metric define a norm?

**Definition.** Given a space $X$ with metric $d$, an open ball centered at $x$ of radius $\varepsilon$ is the set $B_\varepsilon(x) = \{ y \in X : d(x, y) < \varepsilon \}$.

**Problem 4.** Given a space $X$ with the discrete metric $\rho$, prove the following statements.

(a) For any $x \in X$, the set $\{x\}$ is an open set.
(b) All sets in $X$ are open.
(c) All sets in $X$ are closed.
(d) Any subset $A \subseteq X$, such that $|A| \geq 2$ is disconnected.

*Note:* $(X, \rho)$ has the largest possible topology, i.e., all subsets of $X$ are open. This is also referred to as discrete topology. On the other hand, trivial topology is a topology which only has two open sets: the empty set and the whole space.

Let $X$ be a space. Define a function $s : X \times X \to \mathbb{R}$ by $s(x, y) = 0$ for all $x, y \in X$.

**Problem 5.** Is $s$ a metric? Why or why not?

**Problem 6.** Let $X$ be a space, equipped with the pseudo-metric $s$.

(a) Find all the open sets in $X$.
(b) Find all the closed sets in $X$.
(c) Are there subsets of $X$ which are disconnected? Why?
(d) Let $A \subset X$ be a nonempty proper subset of $X$. Find $A^0$ and $\bar{A}$. 