Ch 3.2: Direct proofs

A direct proof is a way of showing that a given statement is true or false by using existing lemmas and theorems without making any further assumptions. To prove statements of the form "if P, then Q",

Assume that **the statement P is true** and directly derive the conclusion that **the statement Q is true**.

We can use the following properties of integers without justification.

- The sum (difference, product) of every two integers is an integer.
- The product of two negative integer is positive.
- Every integer is of the form 2m or 2m+1, where $m \in \mathbb{Z}$.

$\vdots$

Definition: An integer x is called even (respectively odd) if there is an integer k for which x = 2k (respectively 2k+1).

Example. If n is an even integer, then $7n + 4$ is also an even integer.

Write a *hypothesis* and a *conclusion* first and fill out *the body of the proof* which is a bridge of logical deductions from the hypothesis to the conclusion.

Proof.

Exercises

1. Let $n \in \mathbb{Z}$. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

2. Let $S = \{0, 1, 2\}$ and let $n \in S$. Prove that if $(n+1)^2(n+2)^2/4$ is even, then $(n+2)^2(n+3)^2/4$ is even.

Proofs Involving Inequalities

(A1) For all real numbers a, b, c, if $a \leq b$ and $b \leq c$ then $a \leq c$.

(A2) For all real numbers a, b, c, if $a \leq b$ then $a + c \leq b + c$.

(A3) For all real numbers a, b, c, if $a \leq b$ and $0 \leq c$ then $ac \leq bc$.

Prove the statements below using A1-A3, together with any basic facts about *equality* $=$.

(1) For all real numbers a, b, if $0 \leq a$ and $a \leq b$ then $a^2 \leq b^2$.

(2) For all real numbers a, if $a \leq 0$ then $0 \leq -a$.

Working Backwards

Theorem (Inequality between arithmetic and geometric mean.)

$$\text{If } a, b \in \mathbb{R} \text{ are such that } a \geq 0 \text{ and } b \geq 0, \text{ then } \frac{a+b}{2} \geq \sqrt{ab}$$

Scratch work:

1. Start with the inequality you are asked to prove.
2. Simplify it as much as possible until you arrive at a statement that is obviously true.

Formal Proof:

3. In order to write formal proof, now start from the obviously true statement.
4. Use your previous work to guide you on how to arrive at the desired inequality.

Proof:

Common mistakes.

1. **What is wrong with this proof?**
   (1) Assume $a = b$.
   (2) Multiplying both sides by b, $ab = b^2$.
   (3) Subtracting a^2 from both sides, $ab - a^2 = b^2 - a^2$.
   (4) Factoring $a(b - a) = (b + a)(b - a)$.
   (5) Dividing by $b - a$, $a = b + a$.
   (6) Using (1), $a = 2a$.
   (7) Dividing by a, $1 = 2$.

2. **Circular argument** Prove if n^3 is even then n is even.

   Proof:

   Assume n^3 is even.
   Then $\exists k \in \mathbb{Z}$ such that $n^3 = 8k^3$.
   It follows that $n = \left(8k^3\right)^{1/3} = 2k$.
   Therefore n is even.

   All statements in the proof are true but is the proof correct?

Ch 3.3: Proof by contrapositive

It is a direct proof but we start with the contrapositive because

$$P \implies Q \text{ is equivalent to } \sim(Q) \implies \sim(P).$$

Why do we prove the contrapositive of the implication instead of the original implication?

Example. Prove: If n^3 is even then n is even.

Definition: Two integers are said to have the same parity if they are both odd or both even.

Theorem: Let $x, y \in \mathbb{Z}$. Then x and y are of the same parity if and only if $x + y$ is even.

Proof.

Exercises

1. Prove that if $5x - 11$ is an even integer, then x is an odd integer.

2. Prove that if $7x + 4$ is even, then $5x - 11$ is odd.
(This problem is a continuation of the previous question. Of course we can prove this statement directly by using the definition of an even and odd integers)

3. Let $\{A, B\}$ be a partition of the set of $S = \{1, 2, \cdots, 7\}$, where $A = \{1, 4, 5\}$. Let $n \in S$. Find the set B and prove that if $(n^2 + 3n - 4)/2$ is even, then $n \in A$.

Ch 3.4: Proof by cases

Let a problem of the form

$$(P_1 \text{ or } P_2 \text{ or } \cdots \text{ or } P_n) \implies Q$$

be given, where $P_1, P_2, \cdots, P_n$ are cases.

Show that it is equivalent to the following :

$$(P_1 \implies Q) \text{ and } (P_2 \implies Q) \text{ and } \cdots \text{ and } (P_n \implies Q).$$

Thus, we need to prove all the clauses are true.

Example: Prove that if $n \in \mathbb{Z}$, then $n^3 - n$ is even.

Example: Prove "$n^2 \geq n$ for any integer n".

Proof:

1. How many cases do we need to consider?

2. Explicitly state what we want to prove.

3. If we can prove that all cases are true, we can conclude that $n^2 \geq n$ for all integers n.

4. Write the final polished form of your proof.

Exercises

1. For $n \in \mathbb{Z}$, prove that $9n^2 + 3n - 2$ is even.

2. For any integer n, $n^3 + n$ is an even integer.

3. For $x, y \in \mathbb{R}$, $|xy| = |x||y|$.

4. Prove that for all $x \in \mathbb{R}$, $-5 \leq |x + 2| - |x - 3| \leq 5$.

5. Prove that if x is a real number such that $\dfrac{x^2 - 1}{x + 2} > 0$, then $x > 1$ or $-2 < x < -1$.

More exercise problems for Chapter 3

Section 3.2

3.2.A Let $n \in \mathbb{N}$. Prove that $1^2 + 2^2 + \cdots + n^2$ is even if n is an integer of the form $n = 4k$ for some integer k.

3.2.B Show that if n is an even integer then either $n = 4k$ or $n = 4k+2$ for some integer k. (Hint: For n to be even means that $n = 2m$ for some integer m. Consider two possibilities for m.)

3.2.C Let $n \in \mathbb{N}$. Prove that $1^2 + 2^2 + \cdots + n^2$ is odd if n is an even integer and $n \neq 4k$ for any integer k.

Section 3.3

3.3.A Prove the following statements by stating and proving the contrapositive

(a) If n^2 is an odd integer, then n is an odd integer.

(b) If n^2 is divisible by 4, then n is even.

(c) Let a and b be nonnegative real numbers. If $a^2 < b^2$, then $a < b$. (Hint: Use the following property of the real numbers: if $a < b$ and $c > 0$, then $ac < bc$.)

(d) Let a and b be nonnegative real numbers and let $n \in \mathbb{N}$. If $a^n < b^n$, then $a < b$.

Section 3.4

3.4.A Use a proof by cases to prove that every integer is of the form $4k$, $4k + 1$, $4k + 2$, or $4k + 3$ for some integer k. (Hint: this is similar to 3.2.B above.)

3.4.B Use a proof by cases to prove that if $n = m^2$ for some integer m, then $n = 4k$ or $n = 4k + 1$ for some integer k.