2.9: Some Fundamental Properties of Logical Equivalence

Theorem: For statements P, Q, and R, the following properties hold.

1. Commutative Laws

   (1) $P \vee Q \equiv Q \vee P$
   (2) $P \wedge Q \equiv Q \wedge P$

2. Associate Laws

   (1) $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
   (2) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$

3. Distributive Laws

   (1) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
   (2) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

4. De Morgan's Laws

   (1) $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$
   (2) $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$

Let us prove 3–(2) and 4–(1) and we leave the remaining parts to do on your own. Prove the logical equivalence using truth tables.

Proof of 3–(2) :

P	Q	R					

Proof of 4–(1):

P	Q					

Prove that $\Big(P \Rightarrow Q\Big) \equiv \Big((\sim P) \vee Q\Big)$.

P	Q			

The negation of an implication

Using the fact that $\Big(P \Rightarrow Q\Big) \equiv \Big((\sim P) \vee Q\Big)$ and **De Morgan's Laws**, prove

$$\Big(\sim (P \Rightarrow Q)\Big) \equiv \Big(P \wedge (\sim Q)\Big).$$

Proof:

Remark:

- We already proved the logical equivalence of the two statements, $\sim (P \Rightarrow Q)$ and $P \wedge (\sim Q)$, by using a truth table.
- This class is a writing class. Be careful with all notations!!!
  "*Equal*" means equal, that is, "=" is different from "$\equiv$".

Example: "If you are an athletic student, you must take MTH299".
If this is not true, then it follows that

The negation of a biconditional

Using the fact that $\left(P \iff Q\right) \equiv \left((P \implies Q) \wedge (Q \implies P)\right)$ and **De Morgan's Laws**, prove

$$\sim (P \iff Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P)) \, .$$

Proof:

Remark:

- Of course, we can prove the logical equivalence of the two statements, $\sim (P \iff Q)$ and $(P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$ by using a truth table.

Example: "If you earned a score of 800 on the math SAT, then you will receive a scholarship, and this is the only way you can receive a scholarship. " If this is not true, then it follows that

2.10: Quantified statements

Definition (**Universal quantifier**): $\forall$ - *for all* or *for each* or *for every*

- "$\forall x \in S,\ P(x)$" : For every $x \in S$, $P(x)$.
  - (i) "$\forall x \in S,\ P(x)$" is true if $P(x)$ is true **for every** $x \in S$.
  - (ii) "$\forall x \in S,\ P(x)$" is false if there exist **at least one** element $x \in S$ such that $P(x)$ is false.

- Express each of the following statements using quantifiers.
  - (i) Every student who registers for MTH299 must take the first midterm exam on February 13, 2014.
  - (ii) The product of any two rational numbers is a rational number.
  - (iii) For all real numbers x, x^2 is nonnegative.

- Express each of the following statements in words.
  - (i) $\forall x, y \in \mathbb{N}, \quad x + y \in \mathbb{N}$.
  - (ii) If $a \in A \Rightarrow a \in B$, $\forall a \in A$, then $A \subseteq B$.
  - (iii) $\forall a, b \in A = \{x \mid x = 2n + 1, n \in \mathbb{Z}\}$, $ab \in A$.

Definition (**Existential quantifier**): $\exists$ - *there exists* or *there exist* or *for some*

- $\exists x \in S,\ P(x)$: There exists $x \in S$ such that $P(x)$.

- Express each of the following statements using quantifiers.

  (i) There is a student who didn't turn in an essay homework.

  (ii) There is no student who skipped the first class.

  (iii) Rolle's Thm.: Let f be continuous on $[a, b]$ and differentiable on (a, b).
  If $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

  (iv) The equation $x^2 - 2y^2 = 3$ has an integer solution.

- Express each of the following statements in words.

  (i) f continuous on $[a, b]$ and $f(a) \cdot f(b) < 0 \Rightarrow \exists c \in (a, b)$ such that $f(c) = 0$.

  (ii) $\exists x \in A \Rightarrow A \neq \emptyset$.

  (iii) $\exists x \in \mathbb{Z},\ x^2 = x$.

Combining the quantifiers
Express each of the following statements using quantifiers.

1. There exists a smallest natural number.

2. Every integer is the product of two integers.

3. There exists no smallest integer.

4. For all real number x, there exists a real number y such that $y = x^2$.

5. For any natural number x, there exists an even prime number p larger than x.

6. The set B has a lower bound. (That is, there exists a value b such that b is less than or equal to every element x in B.)

Order of $\exists$ and $\forall$ matters!
In each of the following cases explain what is meant by the statement and decide whether it is true or false.

1. $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ such that $x + y = 1$.

2. $\exists y \in \mathbb{Z}$ such that $\forall x \in \mathbb{Z}$, $x + y = 1$.

3. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ such that $xy = x$.

4. $\exists y \in \mathbb{Z}$ such that $\forall x \in \mathbb{Z}$, $xy = x$.

5. $\lim_{x \to c} f(x) = L$ if $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that $0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.

6. $\lim_{x \to c} f(x) = L$ if $\exists \delta > 0 \quad \forall \varepsilon > 0$ such that $0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.

7. $f : A \to B$ is surjective provided $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.

Existence and Uniqueness: $\exists!$

A symbol $\exists!$

- "$\exists! n$ such that ... " means

(i) **Existence** ($\exists n$ such that . . .)

(ii) **Uniqueness**
(if n_1 and n_2 both have the given property, then $n_1 = n_2$).

Examples
Prove the following: (*we will revisit these problems in Ch.5*)

1. $\forall x \in \mathbb{Q} \setminus \{0\}$, $\exists! y \in \mathbb{Q}$ such that $xy = 1$.

2. Let $f : [0,1] \to \mathbb{R}$ be continuous and strictly increasing with $f(0) \cdot f(1) < 0$. Then $\exists! x \in [0,1]$ such that $f(x) = 0$.

Examples: Express in terms of quantifiers

(i) There are at least two black kittens.

(ii) There are at least three people who do not like ice cream.

How can you generalize the above strategy?

(iii) There are at least n nonzero elements in the set A.

(iv) For every element in the set B there are at least n elements in the set A similar to it.

Interpreting Multiple Quantifiers

Examples Are these statements true or false? Prove your claim. If the statement is not true, provide a *counterexample* to prove your claim.

1. $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ such that $x + y = 1$.

2. $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ such that $xy = 1$.

3. $\exists y \in \mathbb{Z}$ such that $\forall x \in \mathbb{Z}$, $xy = x$.

Discuss what the following are actually saying:

4. $\forall x \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists q \in \mathbb{Q}$ such that $|x - q| < \varepsilon$.

5. $\forall x \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists q \in \mathbb{N}$ such that $|x - q| < \varepsilon$.

Negation of $\exists$ and $\forall$:

What is the negation of the following statements?

1. "**All horses are white**."

2. "**There were some students who missed class last Friday**."

3. $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ such that $x + y = 1$.

4. $\exists y \in \mathbb{Z}$ such that $\forall x \in \mathbb{Z}$, $xy = x$.

Theorem 1

$$\sim (\forall x,\, P(x)) \quad \equiv \quad \exists x,\, \sim (P(x))$$

Theorem 2

$$\sim (\exists x,\, P(x)) \quad \equiv \quad \forall x,\, \sim (P(x))$$

$\star$ What would be the negation of $\exists x \forall y \exists z, P(x,y,x)$?

To negate a general statement with quantifiers:

(a) Maintaining the order of the quantified segments, change each ($\forall$...) segment into a ($\exists$... such that) segment;

(b) Change each ($\exists$... such that) segment into a ($\forall$...) segment;

(c) Negate the final statement

Examples: Negate the following statements:

1. $\exists y \in \mathbb{Z}$ such that $\forall x \in \mathbb{Z}$, $x + y = 1$.

2. $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ such that $xy = x$.

3. $\lim\limits_{x \to c} f(x) = L$ if $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that $0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$. *This is a formal definition of continuity of a function f at a point $x = c$.*

4. $\lim\limits_{n \to \infty} a_n = L$ if $\forall \varepsilon > 0 \quad \exists N \in \mathbb{R}$ such that $\forall n > N$, $|a_n - L| < \varepsilon$. *This is a formal definition of the limit of sequences. We will revisit it in Ch.12.1*

5. $f : A \to B$ is surjective provided $\forall y \in B\, \exists x \in A$ such that $f(x) = y$. *This is a formal definition of a surjective function. We will revisit it in Ch.9.3*

Example: Prove that $\forall x \in \mathbb{N}$, $\nexists y \in \mathbb{R} \setminus \{x\}$ such that y is closest to x.

Equivalent to: $\forall x \in \mathbb{N}$,

$$\text{not}(\exists y \in \mathbb{R} \setminus \{x\} \text{ such that } \forall z \in \mathbb{R} \setminus \{x\}, |x - y| < |x - z|)$$

If we restate the previous sentence without "not", that is, we negate the previous sentence, we need to show that
$\forall x \in \mathbb{N}$,

$$\forall y \in \mathbb{R} \setminus \{x\}\ \exists z \in \mathbb{R} \setminus \{x\} \text{ such that } \mid x - y \mid > \mid x - z \mid$$

Proof:

Negation of $\exists!$

- (Review) The notation $\exists! n$ indicates

  (i) Existence ($\exists n$ such that . . .)

  AND

  (ii) Uniqueness
  (if $n_1 \in Z$ and $n_2 \in Z$ both have the given property, then $n_1 = n_2$).

By using the De Morgan's Law "$\sim (P \wedge Q) \equiv (\sim (P) \vee \sim (Q))$"

- Negation of unique existence: "not $\exists!$"

  (i) There does not exist n such that . . .

  OR

  (ii) It is not unique

Examples Negate the following

1. $\forall x \in \mathbb{Q} \setminus \{0\}$, $\exists! y \in \mathbb{Q}$ such that $xy = 1$.

2. Let $f : [0, 1] \to$ be continuous and strictly decreasing with $f(0) \cdot f(1) < 0$. Then $\exists! x \in [0, 1]$ such that $f(x) = 0$.