Math 299 | **Lecture 24: Section 9.6**
Inverse Functions

Invertible functions

A function $f : A \to B$ is **invertible** if
$f^{-1} : B \to A$ is a function.

$\star$ What is the definition of a function?

- First, every element in a domain has an image in a codomain.
- Second, the image of each element in a domain is unique!!

Theorem: Let $f : A \to B$ be a function.
f is bijective if and only if f is invertible and the inverse function is also bijective.

Remark: Let f be bijective. Then both composition functions are well defined by $(f^{-1} \circ f)(x) = x$ for all $x \in X$ and $(f \circ f^{-1})(y) = y$ for all $y \in Y$.

Example Let the set $S = \{$ all 50 states. $\} = \{$AL, AK, $\cdots$, WY $\}$, and the set $C = \{$ all US cities. $\}$. Define the functions $f : S \to C$ and $g : C \to S$ with

$$\begin{aligned} f(\text{state}) &= \text{its capital city} \\ g(\text{city}) &= \text{the state it is in.} \end{aligned}$$

For example, $f(\text{MI}) = \text{Lansing}$, and $g(\text{Detroit}) = \text{MI}$.

1. Determine whether f and g are injective, surjectie, and/or bijective. Explain very briefly.
2. Prove that, for all $s \in S$, we have $g(f(s)) = s$.
3. It is false that $f(g(c)) = c$ for all $c \in C$. Find a counter–example, that is, an example of c where the equation is false.
4. Explain how to take a smaller set of cities $\overline{C} \subset C$ to get new, bijective functions : $\overline{f} : S \to \overline{C}$ and $\overline{g} : \overline{C} \to S$ having different domain or codomain, but defined by the same rules. Prove that $\overline{f}$ and $\overline{g}$ are inverse functions.

Exercises

1. Let $A = \mathbb{R} - \{1\}$ and define $f : A \to A$ by $f(x) = \frac{x}{x-1}$ for all $x \in A$.
   (a) Prove that f is bijective.
   (b) Determine f^{-1}.
   (c) Determine $f \circ f \circ f$.

2. Let $A = \{x | x \in \mathbb{R} \text{ and } x > 0\}$. The function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - 4x + 5$. What is the largest codomain so that f is surjective?

3. Let $A = \{x | x \in \mathbb{R} \text{ and } x \geq 2\}$ and $B = \{x | x \in \mathbb{R} \text{ and } x \geq 1\}$ and the function $f : A \to B$ is defined by $f(x) = x^2 - 4x + 5$. If there exists an inverse function f^{-1}, then find the inverse function of the function f and specify the domain and codomain of the inverse function.