Bijective functions

**Definition:** A function \( f : X \to Y \) is **bijective** (or **one–to–one correspondence**) if \( f \) is both injective and surjective.

**Theorem.** If \( A \) and \( B \) are finite sets with \( |A| = |B| = n \), then there are \( n! \) bijective functions from \( A \) to \( B \).

**Theorem.** Let \( A \) and \( B \) be finite nonempty sets with \( |A| = |B| \) and let \( f \) be a function from \( A \) to \( B \). Then \( f \) is one-to-one if and only if \( f \) is onto.

• Does this hold if \( A \) and \( B \) are infinite sets?
Examples:

1. Prove that the function $f : \mathbb{R} - \{5\} \to \mathbb{R} - \{1\}$ defined by $f(x) = \frac{x}{x - 5}$ is bijective.

2. Prove that the function $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ defined by $f([x]) = [5x + 2]$ is a well defined bijective function.

Composition of Functions

Definition: If $f : A \to B$ and $g : B \to C$ are functions, then $g \circ f$ is a function from $A$ to $C$ defined by $(g \circ f)(x) = g(f(x))$. It is called the composition of $f$ and $g$.

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 2x + 5$ and $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. Find the domain and range of $f$ and $g$, as well as $f \circ g$ and $g \circ f$ (where they are defined).
2. \( f = \{(1, m), (2, n), (3, m)\} \), \( g = \{(k, 1), (l, 2), (m, 1), (n, 3)\} \). Find \( f \circ g \) and \( g \circ f \).

**Theorem.** Let \( f : A \to B \) and \( g : B \to C \) be two functions.

(a) If \( f \) and \( g \) are injective, then so is \( g \circ f \).

(b) If \( f \) and \( g \) are surjective, then so is \( g \circ f \).

**Corollary.** Let \( f : A \to B \) and \( g : B \to C \) be bijective functions, then so is \( g \circ f \) is bijective.