Math 299 **Lecture 23: Sections 9.1 -9.3**
Functions. Injective and Surjective Functions.

Maps and functions: Definitions

Definition: A *function* or *map* from D to R is a triplet of objects $< f, D, R >$, where D and R are sets and f is a subset of $D \times R$ with the following two properties:

(a) For all $x \in D$, there exists an element y in R such that $(x, y) \in f$.

(b) If $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.

1. If every element of D *is related to* **exactly** one element of R, then f is a function from D to R. (This is the most studied type of relation.)

2. The set D is called the **domain** of f.

3. The set R is called the **codomain** of f.

4. The set $\{y|(x, y) \in f\}$ is called the **range** of f.

5. If $(x, y) \in f$, y is the image of x with respect to f. It is called the value of x under f and is denoted by $f(x)$, that is, $y = f(x)$.

6. If $(a, b) \in f$, then b is refered as the **image** of a and is denoted by $b = f(a)$.

7. If A is a subset of D, the **image** $f(A)$ of A is defined as
$$f(A) = \{f(x) \in R|x \in A\}.$$
Thus, $f(C)$ is a subset of R.

8. For a subset B of R, the **inverse image** $f^{-1}(B)$ of B is defined as
$$f^{-1}(B) = \{x \in D|f(x) \in B\}.$$
Thus, $f^{-1}(B)$ is a subset of D.

The Set of All Functions from A to B. For nonempty sets A and B, the set of all functions from A to B is denoted by B^A, i.e., $B^A = \{f \mid f : A \to B\}$.

EX. List all possible functions f from $\{a, b\}$ to $\{1, 2, 3\}$.

For **finite** sets A and B, the number of functions from a set A to a set B is

$$\left|B^A\right| = |B|^{|A|}.$$

Injective functions. A function $f : X \to Y$ is **injective** (or **one–to–one**) if

$$\text{for all } x_1, x_2 \in X, \ f(x_1) = f(x_2) \text{ implies } x_1 = x_2.$$

- The definition states that any element of a set Y has *no more than one* pre–image.
- To prove the injectivity of a function, suppose that $f(x_1) = f(x_2)$ and show by direct implications that $x_1 = x_2$.
- Since every two elements of A must have **distinct** images in B, if X and Y are finite sets, $|X| \leq |Y|$.
- We can intuitively test whether a function is injective by using the horizontal line test. Why?
- However, in this class we should rigorously prove it by using the definition and the horizontal line test is not accepted as a formal proof.

Examples

A. Consider the function $f : \{a, b, c\} \rightarrow \{2, 4, 6, 8\}$ defined by $f(a) = 4$, $f(b) = 2$, and $f(c) = 8$. Is this function injective?

B. Let $A = \{x | x \in \mathbb{R} \text{ and } x \neq 2\}$ and the function $f : A \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{4x}{x-2}$. Prove that the function f is injective.
Solution:

C. Let the function $f : [-1, 1] \rightarrow [0, 1]$ be defined by $f(x) = |x|$.
Is this function injective?
How about if its domain is $[0, 1]$, instead of $[-1, 1]$?

Exercises

1. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + 1$ is injective.

2. Show that the function $f : \mathbb{N} \to \mathbb{R}$ given by $f(x) = x^2$ is injective.

3. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x + y, x - y)$ is injective.

4. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - x$ is injective.

5. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not injective.

Surjective functions. A function $f : X \to Y$ is **surjective** if

$$\text{for all } y \in Y, \text{ there exists } x \in X \text{ such that } f(x) = y.$$

- The definition states that every element of Y is the image of *at least* one element of a set X. Thus, if X and Y are finite sets, $|Y| \leq |X|$.
- Note that, by the definition of a surjective function, $Y \subseteq \text{Range}(f)$. Since $\text{Range}(f) \subseteq Y$, $\text{Range}(f) = Y$.
- Proving surjectivity:

  If any value y in Y is given,
  we have to prove the existence of x in X such that $f(x) = y$.

  The procedure for proving that a function is surjective is :

  1. Let y be in the codomain such that $y = f(x)$.
  2. Solve the equation $y = f(x)$ for x.
  3. Check that x found in the second step is an element in the domain of f.

Examples

- Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is surjective.

Exercises

1. If the function $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$f(x) = \begin{cases} x^2 & \text{if } x > 0 \\ x & \text{if } x \leq 0, \end{cases}$$
prove that the function f is surjective.

2. If the function $f : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ defined by $f(x) = |x|$, prove that f is a surjective function.

3. If the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x, y) = (x + y, x - y)$, prove or disprove that the function f is surjective.

4. Let $A = \{x | x \in \mathbb{R} \text{ and } 0 \leq x \leq 5\}$ and $B = \{x | x \in \mathbb{R} \text{ and } 2 \leq x \leq 8\}$. The function $f : A \to B$ is defined by $f(x) = x + 2$. Prove that the function f is not a surjective function.