
Definition:

A **set** is a well–defined collection of distinct objects called the **elements** (or **members**).

Sets are conventionally denoted with capital letters.

1. We enclose a set with braces (curly brackets) $\{\}$.
2. The objects can be anything.
3. Order of elements is irrelevant and no repeated elements are allowed.
4. We can describe a set in two different ways
   (a) An extensional definition: simply list elements in curly brackets:
   Ex: A=$\{$a, MATH, 3$\}$, B=$\{1, 2, 3, 4, \cdots\}$, $C = \{\phi, Ah, \{1, 2\}\}$.

   (b) An intensional definition: use a rule.
   Ex: Let $S(x)$ be a statement.
   - C=$\{x \mid S(x)\}$ asserts that
     C is the class all the elements x which satisfy $S(x)$.

   - B=$\{x \mid x$ is even positive number$\}$.

Examples.

1. Write the following sets by listing its elements.
   (1) $S_1 = \{x \mid |x| < 5 \text{ and } x \text{ is a negative integer}\}$

   (2) $S_2 = \{x \mid x^2 = 7\}$

   (Unless otherwise specified, sets are assumed to be a subsets of the real numbers.)

2. Write the following sets using an intensional definition (using a rule).
   (1) $S_3 = \{..., -6, -4, -2, 0, 2, 4, 6, ...\}$

   (2) $S_4 = \{1, 8, 27, 64, 125\}$

We will use the following notational convention:

(a) Lower–case letters $a, b, x, \ldots$ will be used only to designate elements.

(b) A capital letter may denote either an element or a set.

1. $x \in X$: x is an element of a set X.

2. $x \notin X$: x is not an element of a set X.

3. **Definition**: The set with no element is called the **empty set** and we denote it by $\phi = \{\}$.

4. **Definition**: A universal set $\mathcal{U}$ is the set of **all** elements in the domain of discourse. In our previous examples, this is the set of real numbers.

5. **Definition**: If a set A has a finite number of elements, then A is a **finite set**. The number of elements of a set A is called the **cardinality** of A and we write $|A|$.

Examples. Find the cardinalities of the following sets.

(a) $\emptyset$

(b) $\{\emptyset\}$

(c) $\{\{1, 2\}, 3, 4\}$

6. **Definition**: Let A and B be sets. If **every** element of A is an element of B, a set A is a <u>subset</u> of a set B.

   This is denoted by $A \subseteq B$.
   This means that, if $c \in A$, then $c \in B$.

7. **Definition** If A is a subset of B and A is not equal to B we denote this by $A \subset B$. A is called a <u>proper subset</u> of B.

8. **Definition**: We define $A = B$ to mean that every element of A is an element of B and vice versa.
   That is, if $x \in A$ implies $x \in B$ **and** $x \in B$ implies $x \in A$, then A is equal to B.

9. **Definition**: The set consisting of all subsets of a given set A is called the power set of A and is denoted by $\mathcal{P}(A)$.

Examples

A. Given the set $S = \{\{1, 2\}, 3, 4\}$.

   (a) List the elements of S.

   (b) Which of the following are true statements?

      (i) $2 \in S$

      (ii) $\{1, 2\} \in S$

      (iii) $\{1, 2\} \subseteq S$

      (iv) $\{3, 4\} \subseteq S$

B. Find the corresponding powersets of the set $M = \{0, 1\}$ and of the set $K = \{a, b, c\}$.

C. What is the cardinality of $\mathcal{P}(M)$? What is $|\mathcal{P}(K)|$?

D. Can you make a conjecture about how A and $|\mathcal{P}(A)|$ are related if A is a finite set?

1. **Theorem**: For any set A,

    I. $\phi \subseteq A$.

    II. $A \subseteq \mathcal{U}$.

2. **Theorem**: For any sets A, B and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Some interesting sets of numbers

- **Natural numbers**: denoted by $\mathbb{N}$.

$$\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\}$$

Note: To remember that $0 \in \mathbb{N}$*, think of the letter 'N' as an abbreviation for the word 'nonnegative'. Not all authors follow this convention.*

- **Integers**: denoted by $\mathbb{Z}$.

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, \cdots\}$$

Note: The letter 'Z' comes from the German word 'zahlen' which means 'to count'.

- **Rational numbers**: denoted by $\mathbb{Q}$.

$$\mathbb{Q} = \{x \mid x \text{ can be written in the form } \frac{p}{q} \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0.\}$$

Note: A rational number x can be expressed in more than one way as a fraction p/q of two integers. For example, if $x = 1$, then x can be written as $1/1$ *or as* $2/2$.

- **Real numbers**: denoted by $\mathbb{R}$.

It is tricky to give a correct definition since we have assumed $\mathbb{R}$ to be our universal set in the above discussion. Here is a description worth thinking about: the real numbers consists of values corresponding to points on a continuous line which extends infinitely in both directions. A point (the *origin*) is chosen on the line and the line is given a sense of 'to the right of the origin' and 'to the left of the origin'. The value 0 corresponds to the origin. The positive real numbers correspond to points to the right of the origin. One point to the right of the origin is selected to represent 1. From this it is clear which points correspond to the integers. Which points correspond to the rational numbers? Does every point correspond to a rational number? These last two questions are meant to be pondered rather than answered immediately. You may revisit this foundational question again in a course on mathematical analysis. We will revisit the second question later in this course.

- **Complex numbers**: denoted by $\mathbb{C}$.

$$\mathbb{C} = \{x \mid x \text{ can be written in the form } a + bi, \text{ where } a, b \in \mathbb{R} \text{ and } i^2 = -1.\}$$

Note: Two complex numbers, $a+bi$ and $c+d\,i$, represent the same element if an only if $a = c$ and $b = d$. Complex numbers of the form $a + 0i$ are denoted by the single value a, and a is viewed as a real number contained in the set of complex numbers. Complex numbers of the form $0+bi$ are denoted by bi; such complex numbers are said to be purely imaginary.

Remark: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Operations on sets:

All logical relations between a finite collection of sets can be represented graphically by means of a **Venn diagram**. A set is represented by a simple (closed) plane area.

- The **union** of A and B is the set of all the elements which are in either A or B or in both. It is denoted by $A \cup B$ and in symbols,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

- The **intersection** of A and B is the set of all elements which are in A and B. It is denoted by $A \cap B$ and in symbols,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

- **Definition**: The **difference** of A and B is the set of elements that are in A but not in B. We denote this by $A - B$ (or $A \setminus B$).

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

- **Definition**: The **complement** of a set A is the set of all the elements which do not belong to A. It is denoted by $\overline{A}$ (or A' or A^c) and in symbols,

$$\overline{A} = \{x \mid x \notin A\}.$$

  or equivalently

$$\overline{A} = \mathcal{U} - A = \{x \mid x \in \mathcal{U} \text{ and } x \notin A\}.$$

Remark: $A - B = A \cap \overline{B}$. Why?

Example. A group incoming first–year students at Michigan State University were surveyed in order to determine the factors that influenced their decision to choose to attend Michigan State University. The survey revealed the following information.

- 55 said ”great football team”.
- 51 said ”very good academic reputation”.
- 61 said ”I was offered financial assistance”.
- 9 said ”great football team” but didn’t say ”very good academic reputation” and didn’t say ”I was offered financial assistance”.
- 26 said ”I was offered financial assistance” and ”great football team” and ”very good academic reputation”.
- 31 said ”very good academic reputation” and ”great football team”.
- 8 said only ”very good academic reputation”.
- 4 said none of the above reasons.

(1) How many said ”I was offered financial assistance” and ”very good academic reputation”?

(2) How many said ”great football team” or ”I was offered financial assistance”?

(3) How many said ”great football team” but not ”very good academic reputation”?

(4) How many were surveyed?

Indexed Collections of Sets

1. **Definition**: The union of $n \geq 2$ sets $A_1, A_2, \cdots, A_n$ is denoted by $\bigcup_{i=1}^{n} A_i$ and is defined as
$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i,\ 1 \leq i \leq n\}.$$

⋆ What does "$a \in \bigcup_{i=1}^{n} A_i$" mean?

2. **Definition**: The intersection of $n \geq 2$ sets $A_1, A_2, \cdots, A_n$ is denoted by $\bigcap_{i=1}^{n} A_i$ and is defined as
$$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for every } i,\ 1 \leq i \leq n\}.$$

⋆ What does "$a \in \bigcap_{i=1}^{n} A_i$" mean?

3. Consider the sets $A_1, A_{10}, A_{11}, A_{12}, A_{31}, A_{42}, \cdots$.

   How can we write the union or intersection of a collection of those sets without listing every set?

   A. Let us call a nonempty set I an **index set**.

   B. Suppose that there is a set S_α where $\alpha \in I$.

   C. $\{S_\alpha\}_{\alpha \in I}$ is the collection of all sets S_α, where $\alpha \in I$, and it is called an **indexed collection of sets**.

Example : $I = \{n \in \mathbb{N} \mid n = 2m+1 \text{ for some } m \in \mathbb{N}\}$ and $S_r = \{r, r+1, r+2\}$. Determine

$$\bigcup_{i \in I} S_i, \quad \bigcap_{x \in I} S_x, \quad \bigcup_{j \in I}^{10} S_j.$$

Partitions of Sets

1. If the intersection of two sets is the empty set, then the two sets are disjoint.

2. Let S be a collection of subsets of a set A.
If every two distinct sets in S are disjoint, then the collection S of subsets of a set A is called pairwise disjoint.

3. Let A be a nonempty set. A **partition** of a set A is defined as a collection S of subsets of A satisfying the three properties

   (1) X is not an empty set for every set $X \in S$.

   (2) For every two sets $X, Y \in S$, either $X = Y$ or $X \cap Y = \{\}$. In other words, S is a collection of pairwise disjoint subsets of A.

   (3) $\displaystyle\bigcup_{X \in S} X = A$

Cartesian Products of Sets

- The order in which the elements of a set A are listed doesn't matter.
- This is not the case for an ordered pair: $(1, 2) \neq (2, 1)$. Note that we use parentheses, not braces, for an ordered pair.
- What does "$(a, b) = (c, d)$" mean?

- How about "$(x_1, x_2, ..., x_n) = (y_1, y_2, ..., y_n)$"? (These are ordered n-tuples).

Definition: Let X and Y be two sets. The **Cartesian product** of X and Y is the set of all possible ordered pairs (x, y) where x belongs to a set X and y belongs to a set Y. We denote it by $X \times Y$.

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

a. Note that $X \times Y$ is not a subset of either X or Y. Why?

b. In general, $X \times Y$ is not equal to $Y \times X$. Why?

c. What is the cardinality of $X \times Y$?

Theorem. $|X \times Y| = |X| \cdot |Y|$.