
Counterexamples

$$\sim (\forall x \in S, P(x)) \equiv \exists x \in S, \sim P(x)$$

If the statement, "$\forall x \in S, P(x)$", is false, there exists $x \in S$ satisfying $\sim P(x)$.

Example

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, (x-1)y = y$$

Is it true or false? Prove or **disprove** a statement.
Examples Prove or disprove each statement.

1. Let a and b be positive real numbers. Then

$$a + b > 2\sqrt{a\,b}.$$

2. There exists a natural number n such that $n^2 + 3n + 2$ is prime.

3. For all real numbers x and y, if $x^2 = y^2$, then $x = y$.

4. (Textbook exercise) For $n \in \mathbb{N}$, if $\frac{n(n+1)}{2}$ is odd, then $\frac{(n+1)(n+2)}{2}$ is odd.

5. Let A, B, and C be sets. Then $A - (B \cap C) = (A - B) \cap (A - C)$.

6. There is a real number x for which $x^4 < x < x^2$.

Proof by contradiction : General

"What does contradiction mean in logic?"

- A contradiction consists of a logical incompatibility between two or more propositions.

  For example, $S \wedge \sim S$ is a contradiction, where S is a statement.

Procedure for proof by contradiction: **Prove a statement S.**

1. Assume that $\sim S$ is true.
2. Follow the logical deductions which will lead to a contradiction.
3. Since the assumption leads to a contradiction, the assumption can't be true. That is, $\sim S$ is false.
4. Since $\sim S$ is false, the statement S is true. (Why?)

Proof by contradiction for implications

Prove that $P \implies Q$ (i.e P is sufficient for Q).

Strategy:

- Assume that the negation of implication, that is, $P \wedge \sim Q$.

  (*we should find* $\neg(Q)$ *first.*)

- Then we will arrive at a contradiction to our assumption "$P \wedge \sim Q$" or something that is well known to be true.
- Therefore, as we've arrived at a false statement, we conclude that our assumption must be wrong.
- Thus $P \wedge \sim Q$ is false.
- Then, by the definition of negation, $P \implies Q$ is true because $P \implies Q$ is equivalent to $\sim (P \wedge \sim Q)$.

Lemma : If n^2 is even, then n is even.

A. Remarks.

(a) Since we will use this result to prove the theorem in next slide we called this a lemma rather than a theorem (or a proposition).

(b) We already proved this lemma by using the contrapositive of the statement and now we will prove it by contradiction.

B. Ingredients for the proof.

(1) What is the negation of the lemma?

(2) What is the assumption if we use the method of proof by the contradiction?

C. Write the proof rigorously with complete sentences.

Theorem : $\sqrt{2}$ is not a rational number.

Exercises

1. The sum of an irrational number and a rational number is irrational.

2. Show that $\log_2 3$ is irrational. Note, by definition $\log_2 3$ is the number x, such that $2^x = 3$.

3. There is no greatest even integer.

4. If $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

5. If $a, b \in \mathbb{Z}$ and $a \geq 2$, then $a \nmid b$ or $a \nmid b + 1$.

6. $\forall x \in \mathbb{R}$, $\big(s + \sqrt{2} \notin \mathbb{Q}\big)$ or $\big(s - \sqrt{2} \notin \mathbb{Q}\big)$.