1. Describe the elements of the set $(\mathbb{Z} \times \mathbb{Q}) \cap \mathbb{R} \times \mathbb{N}$. Is this set countable or uncountable?

   Solution: The set is equal to $\{(x, y) \mid x \in \mathbb{Z}, y \in \mathbb{N}\} = \mathbb{Z} \times \mathbb{N}$. Since the Cartesian product of two denumerable sets is denumerable, this set is denumerable, hence countable.

2. Let $A = \{\emptyset, \{\emptyset\}\}$. What is the cardinality of A? Is $\emptyset \subset A$? Is $\emptyset \in A$? Is $\{\emptyset\} \subset A$? Is $\{\emptyset\} \in A$? Is $\{\emptyset, \{\emptyset\}\} \in A$?

   Solution: $|A| = 2$; it has two elements: $\emptyset$ and $\{\emptyset\}$. The answers to the remaining questions are yes, yes, yes, yes, no.

3. List the elements of the set $A \times B$ where A is the set in the previous question and $B = \{1, 2\}$.

   Solution: $A \times B = \{(\emptyset, 1), (\emptyset, 2), (\{\emptyset\}, 1), (\{\emptyset\}, 2)\}$.

4. Suppose that A, B, and C are sets. Which of the following statements is true for all sets A, B, and C? For each, either prove the statement or give a counterexample: $(A \cap B) \cup C = A \cap (B \cup C)$, $A \cap B \subseteq A \cup B$, if $A \subset B$ then $A \times A \subset A \times B$, $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$.

   Solution: $(A \cap B) \cup C \neq A \cap (B \cup C)$ in general; a counterexample is $A = \{1, 2\}, B = \{1, 3\}, C = \{1, 4\}$. Then $(A \cap B) \cup C = \{1, 4\}$, whereas $A \cap (B \cup C) = \{1\}$.

   $A \cap B \subset A \cup B$ is true. If $x \in A \cap B$, then $x \in A$. So, $x \in A \cup B$.

   $A \subset B \implies A \times A \subset A \times B$ is true. If $(x, y) \in A \times A$, then $x, y \in A$. Therefore, $y \in B$. Therefore, $(x, y) \in A \times B$.

   $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$ is true. Recall that $\cap$ and $\cup$ satisfy associative laws. Thus,

   $$\overline{A} \cap \overline{B} \cap \overline{C} = (\overline{A} \cap \overline{B}) \cap \overline{C} = \overline{A \cup B} \cap \overline{C},$$

   by De Morgan's law. Another application of De Morgan's law yields

   $$\overline{(A \cup B) \cup C} = \overline{A \cup B \cup C}.$$

5. State the negation of each of the following statements:

   - There exists a natural number m such that $m^3 - m$ is not divisible by 3.
   - $\sqrt{3}$ is a rational number.
   - 1 is a negative integer.
   - 57 is a prime number.

6. Verify the following laws:

   - (a) Let P, Q and R are statements. Then,

     $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are logically equivalent.

   - (b) Let P and Q are statements. Then,

     $P \Rightarrow Q$ and $(\sim Q) \Rightarrow (\sim P)$ are logically equivalent.

7. Write the open statement $P(x, y)$: **"for all real x and y the value $(x-1)^2 + (y-3)^2$ is positive"** using quantifiers. Is the quantified statement true or false? Explain.

8. Prove that $3x + 7$ is odd if and only if x is even.

   Solution: First, we will prove that if x is even, then $3x + 7$ is odd. Assume x is even. Then $\exists k \in \mathbb{Z}$ such that $x = 2k$. Therefore, $3x + 7 = 6k + 7 = 2(3k + 3) + 1 = 2s + 1$, where $s = 3k + 3 \in \mathbb{Z}$. Thus, $3x + 7$ is odd. Now, we need to prove that if $3x + 7$ is odd, then x is even. We are going to prove the equivalent, contrapositive statement. Assume x is odd. Then $\exists k \in \mathbb{Z}$ such that $x = 2k + 1$. Therefore, $3x + 7 = 6k + 3 + 7 = 2(3k + 5) = 2s$, where $s = 3k + 5 \in \mathbb{Z}$. Thus, $3x + 7$ is even. Thus, $3x + 7$ is odd if and only if x is even.

9. Prove that if a and b are positive numbers, the $\sqrt{ab} \leq \frac{a+b}{2}$. This is referred to as "Inequality between geometric and arithmetic mean."

   Solution: Let $a, b \in \mathbb{R}^+$. Then $(a - b) \in \mathbb{R}$ and thus $(a - b)^{\geq} 0$. The following inequalities are equivalent.

   $$\begin{aligned} (a-b)^2 &\geq 0 \\ a^2 - 2ab + b^2 &\geq 0 \\ a^2 + 2ab + b^2 &\geq 4ab \\ (a+b)^2 &\geq 4ab \\ a + b &\geq 2\sqrt{ab} \\ \frac{a+b}{2} &\geq \sqrt{ab}. \end{aligned}$$

   Thus, we have arrived at the desired inequality, which holds true for all $a, b \in \mathbb{R}$.

10. Let A, B, and C be sets. Prove that $A \times (B \bigcap C) = (A \times B) \bigcap (A \times C)$.

    Solution: First, we will prove that $A \times (B \bigcap C) \subseteq (A \times B) \bigcap (A \times C)$. Let $(x, y) \in A \times (B \bigcap C)$ be an arbitrary element. Then, $x \in A$ and $y \in B$ and $y \in C$. Thus, $(x, y) \in A \times B$ and $(x, y) \in A \times C$. Therefore, $(x, y) \in (A \times B) \bigcap (A \times C)$. Thus, we can conclude that $A \times (B \bigcap C) \subseteq (A \times B) \bigcap (A \times C)$.

Now, we need to prove that that $(A \times B) \bigcap (A \times C) \subseteq A \times (B \bigcap C)$. Take an arbitrary element $(x, y) \in (A \times B) \bigcap (A \times C)$. Then, $(x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$. Therefore, $x \in A$ and $y \in B$ and $y \in C$. Thus, $y \in B \bigcap C$, which implies $(x, y) \in A \times (B \bigcap C)$.

Since we have proven both inclusions, we can conclude the desired equality of sets, namely, $A \times (B \bigcap C) = (A \times B) \bigcap (A \times C)$.

11. Let A, B, and C be sets. Prove that $(A - B) \bigcap (A - C) = A - (B \bigcup C)$.

12. Suppose that x and y are real numbers. Prove that if $x + y$ is irrational, then x is irrational or y is irrational.

    Solution: We will instead prove the contrapositive statement, which is equivalent to the original one. Assume that $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$. Then $\exists p, q, r, s \in \mathbb{Z}$ such that $x = \frac{p}{q}$ and $y = \frac{r}{s}$. Then $x + y = \dfrac{sp + qr}{sq} \in \mathbb{Z}$. (Alternatively, we can use the fact that Q is closed under addition.) Thus, if x and $y \in \mathbb{Q}$, then $x + y \in \mathbb{Q}$.

13. Let x be an irrational number. Prove that x^4 or x^5 is irrational.

    Solution: We will instead prove the contrapositive statement, which is equivalent to the original one, namely, if x^4 and x^5 are rational, then x is rational. Clearly, if $x^5 = 0$, then $x = 0$, thus this case is trivial. Thus, assume that x^5 and $x^4 \in \mathbb{Q} - \{0\}$. Then $\exists p, q, r, s \in \mathbb{Z} - \{0\}$ such that $x^5 = \frac{p}{q}$ and $x^4 = \frac{r}{s}$. Thus, $x = \dfrac{x^5}{x^4} = \frac{ps}{qr} \in \mathbb{Q}$. This concludes the proof of the contrapositive statement, thus the original statement also holds true.

14. Use a proof by contradiction to prove the following.

    There exist no natural numbers m such that $m^2 + m + 3$ is divisible by 4.

15. Let a, b be distinct primes. Then $\log_a(b)$ is irrational.

16. Prove or disprove the statement: there exists an integer n such that $n^2 - 3 = 2n$.

17. Prove or disprove the statement: there exists a real number x such that $x^4 + 2 = 2x^2$.

18. Prove that there exists a unique real number x such that $x^3 + 2 = 2x$.

19. Disprove that statement: There exists integers a and b such that $a^2 + b^2 \equiv 3 \pmod 4$

20. Use induction to prove that $6|(n^3 + 5n)$ for all $n \geq 0$.

21. Use induction to prove that

    $$1 \cdot 4 + 2 \cdot 7 + \cdots + n(3n + 1) = n(n + 1)^2$$

    for all $n \in \mathbb{N}$.

22. Use the Strong Principle of Mathematical Induction to prove that for each integer $n \geq 11$, there are nonnegative integers x and y such that $n = 4x + 5y$.

23. A sequence $\{a_n\}$ is defined recursively by $a_0 = 1$, $a_1 = -2$ and for $n \geq 1$,

    $$a_{n+1} = 5a_n - 6a_{n-1}.$$

    Prove that for $n \geq 0$,

    $$a_n = 5 \times 2^n - 4 \times 3^n.$$

24. Suppose R is an equivalence relation on a set A. Prove or disprove that R^{-1} is an equivalence relation on A.

    Solution: If R is an equivalence relation, then so is $R^{-1} = \{(y, x) \in A \times A \mid (x, y) \in R\}$.

    Proof 1: Let $a \in A$. Then since R is reflexive we have $(a, a) \in R$. It follows from the definition of R^{-1} that $(a, a) \in R^{-1}$, proving that R^{-1} is reflexive as well. To show that R^{-1} is symmetric, let $(a, b) \in R^{-1}$. Then by definition $(b, a) \in R$. Since R is symmetric, $(a, b) \in R$ as well, and so $(b, a) \in R^{-1}$. To prove that R^{-1} is transitive, let $(a, b), (b, c) \in R^{-1}$. Then $(b, a), (c, b) \in R$, and since R is symmetric, it follows that $(a, b), (b, c) \in R$. By the transitivity of R, we have $(a, c) \in R$ and so $(c, a) \in R^{-1}$. Finally, since R^{-1} is symmetric, it follows that $(a, c) \in R^{-1}$, which shows R^{-1} is transitive.

    Proof 2: We will show that $R = R^{-1}$, and so R^{-1} will automatically be an equivalence relation because we have assumed R is. Let $(a, b) \in R$. Since R is symmetric, $(b, a) \in R$. By the definition of R^{-1} it follows that $(a, b) \in R^{-1}$, which shows $R \subseteq R^{-1}$. The reverse inclusion is similar.

25. Consider the set $A = \{a, b, c, d\}$, and suppose R is an equivalence relation on A. If R contains the elements (a, b) and (b, d), what other elements must it contain?

    Solution: In addition to (a, b) and (b, d), the equivalence relation R must contain

    $$(a, a), (b, b), (c, c), (d, d)$$

$$(b,a),(d,b)$$
$$(a,d)$$
$$(d,a)$$

The elements in the first row appear due to reflexivity; the elements in the second are due to symmetry; the element in the third row is due to transitivity; the element in the last row is due to symmetry from the previous row.

26. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$. Find a relation on $A \times B$ that is transitive and symmetric, but not reflexive.

    Solution 1: Take $R = \emptyset \subset (A \times B) \times (A \times B)$.

    Solution 2: Take $R = \{((a_1, b_1), (a_1, b_1))\}$. This is obviously symmetric (switch (a_1, b_1) with itself), and it is transitive. It is not reflexive because it is missing, say, $((a_2, b_1), (a_2, b_1))$.

    There are many other solutions that are possible. Note that if $((a_i, b_j), (a_k, b_l))$ is in the relation, then so is $((a_k, b_l), (a_i, b_j))$ by symmetry, and hence $((a_i, b_j), (a_i, b_j))$ and $((a_k, b_l), (a_k, b_l))$ are in the relation as well. In particular, to ensure that it is not reflexive, you need to make sure there is at least one element of $A \times B$ that does not appear as a component of any element of the relation.

27. Suppose A is a finite set and R is an equivalence relation on A.

    (a) Prove that $|A| \leq |R|$.
    Solution: Since R is reflexive, if $a \in A$ then $(a, a) \in R$. In particular, the map $f : A \to R$ defined by $f(a) = (a, a)$ is well-defined. This is obviously injective, and so $|A| \leq |R|$.

    (b) If $|A| = |R|$, what can you conclude about R?
    Solution: If $|A| = |R|$ then R contains no more elements than those in the image of f from part (a). This implies that $R = \{(a, a) \mid a \in A\}$ is the diagonal equivalence relation.

28. Consider the relation $R \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by

    $$R = \{(x \bmod 4, 3x \bmod 6) \mid x \in \mathbb{Z}\}.$$

    Prove that R is a function from $\mathbb{Z}_4$ to $\mathbb{Z}_6$. Is R a bijective function?

    Solution: We need to show two things: (1) For every $a \in \mathbb{Z}_4$ there is some $b \in \mathbb{Z}_6$ such that $(a, b) \in R$; (2) If $(a, b), (a, b') \in R$ then $b = b'$. The first follows immediately from the definition of R: if $a = [x] \in \mathbb{Z}_4$, and $x \in [x]$ is any integer, then take b to be the mod 6 reduction of x, and so we have $(a, b) \in \mathbb{Z}_4 \times \mathbb{Z}_6$. To prove (2), suppose $(a, b), (a, b') \in R$. Then we have

    $$(a, b) = (x \bmod 4, 3x \bmod 6),$$

    $$(a, b') = (y \bmod 4, 3y \bmod 6)$$

    for some integers x, y. We obviously have $x \bmod 4 = y \bmod 4$ and so $x = y + 4k$ for some integer k. This gives $3x = 3y + 12k$ and so $b = 3x \pmod 6 = 3y \pmod 6 = b'$, as desired.

29. Consider the relation $S \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by

    $$S = \{(x \bmod 4, 2x \bmod 6) \mid x \in \mathbb{Z}\}.$$

    Prove that S is not a function from $\mathbb{Z}_4$ to $\mathbb{Z}_6$.

    Solution: This fails item (2) in the solution to the previous problem (it satisfies item (1)): We have

    $$0 \pmod 4 = 4 \pmod 4,$$

    but
    $$2 \cdot 0 \pmod 6 \neq 2 \cdot 4 \pmod 6.$$

30. Suppose $f : A \to B$ and $g : X \to Y$ are bijective functions. Define a new function $h : A \times X \to B \times Y$ by $h(a, x) = (f(a), g(x))$. Prove that h is bijective.

    Solution: First we show h is injective. Suppose $h(a, x) = h(a', x')$. Then $f(a) = f(a')$ and $g(x) = g(x')$. Since each of these is injective, it follows that $a = a'$ and $x = x'$, which is equivalent to saying $(a, x) = (a', x')$.

    To see that h is surjective, let $(b, y) \in B \times Y$. Then since f, g are surjective, there are $a \in A$ and $x \in X$ such that $f(a) = b$ and $g(x) = y$. It follows that $h(a, x) = (b, y)$.

31. Prove or disprove: Suppose $f : A \to B$ and $g : B \to C$ are functions. Then $g \circ f$ is bijective if and only if f is injective and g is surjective.

    Solution: The direction ($\Leftarrow$) is false. Indeed, consider the case where $A = B$, and take f to be the identity function (this is obviously injective). Now take g to be any function that is surjective but not injective. Then $g \circ f = g$ is not injective, and so certainly not bijective.

    The direction ($\Rightarrow$) is true. To see this, suppose $g \circ f$ is bijective. If $f(a) = f(a')$, then $(g \circ f)(a) = (g \circ f)(a')$ and so $a = a'$ since $g \circ f$ is injective. To see surjectivity, let $c \in C$. Then since

$g \circ f$ is surjective, it follows that there is some $a \in A$ with $(g \circ f)(a) = c$. Now take $b = f(a)$, and so $g(b) = c$.

32. *(X points)* Let $\mathbb{R}^+$ denote the set of positive real numbers and let A and B be denumerable subsets of $\mathbb{R}^+$. Define $C = \{x \in \mathbb{R} : -x/2 \in B\}$. Show that $A \cup C$ is denumerable.

33. Prove that the interval $(0, 1)$ is numerically equivalent to the interval $(0, +\infty)$.

34. Prove the following statement: A nonemty set S is **countable** if and only if there exists an injective function $g : S \to \mathbb{N}$.

35. Compute the greatest common divisor of 42 adn 13 and then express the greatest common divisor as a linear combination of 42 and 13.

    Solution: $42 = 39 + 3 = 3(13) + 3$;
    $13 = 12 + 1 = 4(3) + 1$; $3 = 3(1) + 0$. Therefore, the gcd is equal to 1. Working backwards, we have that $1 = 13-4(3) = 13-4(42-3(13) = 13(13)+(-4)42$.

36. Let $a, b, c \in \mathbb{Z}$. Prove that if c is a common divisor of a and b, then c divides any linear combination of a and b.

    Solution: Suppose c is a common divisor of a and b and let $ax + by$, where $x, y \in \mathbb{Z}$, be a linear combination of a and b. Then $c \mid a$ and $c \mid b$. Therefore, $a = cm$ and $b = cn$ for some $m, n \in \mathbb{Z}$. It follows that $ax + by = cmx + cny = c(mx + ny)$. Therefore, $c \mid (ax + by)$.

37. Define the term "p is a prime". Then prove that if $a, p \in \mathbb{Z}$, p is prime, and p does not divide a, then $\gcd(a, p) = 1$.

    Solution: A number p is prime if p is a positive integer greater than one and whenever $p = ab$ for some positive integers a and b, then $a = 1$ or $b = 1$.

    Suppose that p is prime and that $a \in \mathbb{Z}$ is not divisible by p. Since p and a are not both zero, there is a greatest common divisor d. If $d > 1$, then $d \mid p$ implies that $d = p$ since the only divisors of p are 1 and p. Since $d \mid a$, this implies that $p \mid a$ which is a contradiction. Therefore, d cannot be greater than 1. Hence, $d = 1$.

38. The greatest common divisor of three integers a, b, c is the largest positive integer which divides all three. We denote this greatest common divisor by $\gcd(a, b, c)$. Assume that a and b are not both zero. Prove the following equation:

    $$\gcd(a, b, c) = \gcd(\gcd(a, b), c).$$

    Solution: Let d be the gcd of a, b, and c. Let e be the gcd of a and b. Let f be the gcd of e and c. We prove that $d = f$. Since e is a linear combination of a and b, $d \mid e$. Since $d \mid c$, and f is a linear combination of e and c, it follows that d divides f. Therefore $d \leq f$.

    On the other hand, $f \mid e$ and $f \mid c$. Since $e \mid a$ and $e \mid b$, $f \mid a$ and $f \mid b$. Thus, f is a common divisor of a, b, and c. Hence, $f \leq d$. Therefore, $f = d$.

39. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove the following:

    $$\lim_{n\to\infty} \frac{3n + 1}{n - 2} = 3.$$

40. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove that

    $$\lim_{n\to\infty} \frac{(-1)^n 3n + 1}{n - 2}$$

    does not exist.

41. Let (a_n) be a sequence with positive terms such that $\lim_{n\to\infty} a_n = 1$. By using the formal definition of the limit of the sequence, prove the following:

    $$\lim_{n\to\infty} \frac{3a_n + 1}{2} = 2.$$

42. (a) Use induction to prove

    $$\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \cdots + \frac{1}{2n(2n + 2)} = \frac{n}{4(n + 1)}$$

    for all $n \in \mathbb{N}$.

    (b) Prove $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k(2k + 2)} = \frac{1}{4}$.