1. Describe the elements of the set \((\mathbb{Z} \times \mathbb{Q}) \cap \mathbb{R} \times \mathbb{N}\). Is this set countable or uncountable?

2. Let \(A = \{\emptyset, \{\emptyset\}\}\). What is the cardinality of \(A\)? Is \(\emptyset \subset A\)? Is \(\emptyset \in A\)? Is \(\{\emptyset\} \in A\)?

3. List the elements of the set \(A \times B\) where \(A\) is the set in the previous question and \(B = \{1, 2\}\).

4. Suppose that \(A, B,\) and \(C\) are sets. Which of the following statements is true for all sets \(A, B,\) and \(C\)? For each, either prove the statement or give a counterexample: 
   - \((A \cap B) \cup C = A \cap (B \cup C)\),
   - \(A \cap B \subseteq A \cup B\),
   - if \(A \subseteq B\) then \(A \times A \subseteq A \times B\),
   - \(A \cap B \cap C = A \cup B \cup C\).

5. State the negation of each of the following statements:
   - There exists a natural number \(m\) such that \(m^3 - m\) is not divisible by 3.
   - \(\sqrt{3}\) is a rational number.
   - 1 is a negative integer.
   - 57 is a prime number.

6. Verify the following laws:
   - (a) Let \(P, Q\) and \(R\) are statements. Then, \(P \land (Q \lor R)\) and \((P \land Q) \lor (P \land R)\) are logically equivalent.
   - (b) Let \(P\) and \(Q\) are statements. Then, \(P \Rightarrow Q\) and \((\sim Q) \Rightarrow (\sim P)\) are logically equivalent.

7. Write the open statement \(P(x, y)\): "for all real \(x\) and \(y\) the value \((x - 1)^2 + (y - 3)^2\) is positive" using quantifiers. Is the quantified statement true or false? Explain.

8. Prove that \(3x + 7\) is odd if and only if \(x\) is even.

9. Prove that if \(a\) and \(b\) are positive numbers, the \(\sqrt{ab} \leq \frac{a + b}{2}\). This is referred to as "Inequality between geometric and arithmetic mean."

10. Let \(A, B,\) and \(C\) be sets. Prove that \(A \times (B \cap C) = (A \times B) \cap (A \times C)\).

11. Let \(A, B,\) and \(C\) be sets. Prove that \((A - B) \cap (A - C) = A - (B \cup C)\).

12. Suppose that \(x\) and \(y\) are real numbers. Prove that if \(x + y\) is irrational, then \(x\) is irrational or \(y\) is irrational.

13. Let \(x\) be an irrational number. Prove that \(x^4\) or \(x^5\) is irrational.

14. Use a proof by contradiction to prove the following.
   - There exist no natural numbers \(m\) such that \(m^2 + m + 3\) is divisible by 4.

15. Let \(a, b\) be distinct primes. Then \(\log_a(b)\) is irrational.

16. Prove or disprove the statement: there exists an integer \(n\) such that \(n^2 - 3 = 2n\).

17. Prove or disprove the statement: there exists a real number \(x\) such that \(x^4 + 2 = 2x^2\).

18. Prove that there exists a unique real number \(x\) such that \(x^3 + 2 = 2x\).

19. Disprove that statement: There exists integers \(a\) and \(b\) such that \(a^2 + b^2 \equiv 3 \pmod{4}\)

20. Use induction to prove that \(6|\left(n^3 + 5n\right)\) for all \(n \geq 0\).

21. Use induction to prove that
   \[1 \cdot 4 + 2 \cdot 7 + \cdots + n(3n + 1) = n(n + 1)^2\]
   for all \(n \in \mathbb{N}\).

22. Use the Strong Principle of Mathematical Induction to prove that for each integer \(n \geq 11\), there are nonnegative integers \(x\) and \(y\) such that \(n = 4x + 5y\).

23. A sequence \(\{a_n\}\) is defined recursively by \(a_0 = 1, a_1 = -2\) and for \(n \geq 1\),
   \[a_{n+1} = 5a_n - 6a_{n-1}\].
   Prove that for \(n \geq 0\),
   \[a_n = 5 \times 2^n - 4 \times 3^n\].

24. Suppose \(R\) is an equivalence relation on a set \(A\). Prove or disprove that \(R^{-1}\) is an equivalence relation on \(A\).

25. Consider the set \(A = \{a, b, c, d\}\), and suppose \(R\) is an equivalence relation on \(A\). If \(R\) contains the elements \((a, b)\) and \((b, d)\), what other elements must it contain?

26. Let \(A = \{a_1, a_2, a_3\}\) and \(B = \{b_1, b_2\}\). Find a relation on \(A \times B\) that is transitive and symmetric, but not reflexive.

27. Suppose \(A\) is a finite set and \(R\) is an equivalence relation on \(A\).
(a) Prove that |A| ≤ |R|.
(b) If |A| = |R|, what can you conclude about R?

28. Consider the relation R ⊂ ℤ₄ × ℤ₆ defined by

R = {(x mod 4, 3x mod 6) | x ∈ ℤ}.

Prove that R is a function from ℤ₄ to ℤ₆. Is R a bijective function?

29. Consider the relation S ⊂ ℤ₄ × ℤ₆ defined by

S = {(x mod 4, 2x mod 6) | x ∈ ℤ}.

Prove that S is not a function from ℤ₄ to ℤ₆.

30. Suppose f : A → B and g : X → Y are bijective functions. Define a new function h : A × X → B × Y by h(a, x) = (f(a), g(x)).

Prove that h is bijective.

31. Prove or disprove: Suppose f : A → B and g : B → C are functions. Then g ∘ f is bijective if and only if f is injective and g is surjective.

32. (X points) Let ℝ⁺ denote the set of positive real numbers and let A and B be denumerable subsets of ℝ⁺. Define C = {x ∈ ℝ : −x/2 ∈ B}. Show that A ∪ C is denumerable.

33. Prove that the interval (0, 1) is numerically equivalent to the interval (0, +∞).

34. Prove the following statement: A nonempty set S is countable if and only if there exists an injective function g : S → ℕ.

35. Compute the greatest common divisor of 42 and 13 and then express the greatest common divisor as a linear combination of 42 and 13.

36. Let a, b, c ∈ ℤ. Prove that if c is a common divisor of a and b, then c divides any linear combination of a and b.

37. Define the term “p is a prime”. Then prove that if a, p ∈ ℤ, p is prime, and p does not divide a, then gcd(a, p) = 1.

38. The greatest common divisor of three integers a, b, c is the largest positive integer which divides all three. We denote this greatest common divisor by gcd(a, b, c). Assume that a and b are not both zero. Prove the following equation:

gcd(a, b, c) = gcd(gcd(a, b), c).

39. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove the following:

\[ \lim_{n \to \infty} \frac{3n + 1}{n - 2} = 3. \]

40. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove that

\[ \lim_{n \to \infty} \frac{(-1)^n 3n + 1}{n - 2} \]

does not exist.

41. Let (aₙ) be a sequence with positive terms such that \( \lim_{n \to \infty} aₙ = 1 \). By using the formal definition of the limit of the sequence, prove the following:

\[ \lim_{n \to \infty} \frac{3aₙ + 1}{2} = 2. \]

42. (a) Use induction to prove

\[ \frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \cdots + \frac{1}{2n(2n + 2)} = \frac{n}{4(n + 1)} \]

for all \( n \in \mathbb{N} \).

(b) Prove \( \sum_{k=1}^{\infty} \frac{1}{2k(2k + 2)} = \frac{1}{4} \).