Consider the following basic facts regarding inequalities.

A1 For all real numbers \(a, b, c\), if \(a \leq b\) and \(b \leq c\) then \(a \leq c\).

A2 For all real numbers \(a, b, c\), if \(a \leq b\) then \(a + c \leq b + c\).

A3 For all real numbers \(a, b, c\), if \(a \leq b\) and \(0 \leq c\) then \(ac \leq bc\).

Prove the statements below using A1-A3, together with any basic facts about equality =.

1. For all real numbers \(a, b\), if \(0 \leq a\) and \(a \leq b\) then \(a^2 \leq b^2\).

2. For all real numbers \(a\), if \(a \leq 0\) then \(0 \leq -a\).

3. For all real numbers \(a, b\), if \(b \leq a\) and \(a \leq 0\), then \(a^2 \leq b^2\).

4. For all real numbers \(b\), \(0 \leq b^2\).

5. For all real numbers \(a, b\), \(ab \leq 1^2(a^2 + b^2)\). \(Hint:\) Consider \((a - b)^2\).

6. For all real numbers \(a, b, \delta\), if \(\delta \neq 0\) then \(ab \leq \frac{1}{2}(\delta^2 a^2 + \delta^{-2}b^2)\).

7. For all real numbers \(a, b\), \(ab = \frac{1}{2}(a^2 + b^2)\) if and only if \(a = b\).

8. For all non-negative real numbers \(a, b\), \(\sqrt{ab} \leq \frac{1}{2}(a + b)\). This is called the arithmetic-geometric mean inequality.

Hints:


2. Use A2 with \(c = -a\).

3. This is similar to Problem 1, except you should use Problem 2 so that A3 applies.

4. Prove using cases: \(0 \leq a\) (use Problem 1) and \(a \leq 0\) (use Problem 3).

5. Use the inequality \(0 \leq (a - b)^2\) (why is this true?) and then FOIL.

6. Notice that \(ab = (\delta a)(\delta^{-1}b)\).