1. Describe the elements of the set \((\mathbb{Z} \times \mathbb{Q}) \cap \mathbb{R} \times \mathbb{N}\). Is this set countable or uncountable?

2. Let \(A = \{\emptyset, \{\emptyset\}\}\). What is the cardinality of \(A\)? Is \(\emptyset \subset A\)? Is \(\emptyset \in A\)? Is \(\{\emptyset\} \subset A\)? Is \(\{\emptyset, \{\emptyset\}\} \in A\)?

3. List the elements of the set \(A \times B\) where \(A\) is the set in the previous question and \(B = \{1, 2\}\).

4. Suppose that \(A, B,\) and \(C\) are sets. Which of the following statements is true for all sets \(A, B,\) and \(C\)? For each, either prove the statement or give a counterexample: \((A \cap B) \cup C = A \cap (B \cup C), A \cap B \subseteq A \cup B,\) if \(A \subset B\) then \(A \times A \subset A \times B,\) \(\overline{A \cap B \cap C} = \overline{A \cup B} \cup C\).

5. State the negation of each of the following statements:
   - There exists a natural number \(m\) such that \(m^3 - m\) is not divisible by 3.
   - \(\sqrt{3}\) is a rational number.
   - 1 is a negative integer.
   - 57 is a prime number.

6. Verify the following laws:
   - (a) Let \(P, Q\) and \(R\) are statements. Then, \(P \land (Q \lor R)\) and \((P \land Q) \lor (P \land R)\) are logically equivalent.
   - (b) Let \(P\) and \(Q\) are statements. Then, \(P \Rightarrow Q\) and \((\sim Q) \Rightarrow (\sim P)\) are logically equivalent.

7. Write the open statement \(P(x, y) : \text{"for all real } x \text{ and } y \text{ the value } (x - 1)^2 + (y - 3)^2 \text{ is positive"}\) using quantifiers. Is the quantified statement true or false? Explain.

8. Prove that \(3x + 7\) is odd if and only if \(x\) is even.

9. Prove that if \(a\) and \(b\) are positive numbers, the \(\sqrt{ab} \leq \frac{a + b}{2}\). This is referred to as “Inequality between geometric and arithmetic mean.”

10. Let \(A, B,\) and \(C\) be sets. Prove that \((B \cap C) = (A \times B) \cap (A \times C)\).

11. Let \(A, B,\) and \(C\) be sets. Prove that \((A - B) \cap (A - C) = A - (B \cup C)\).

12. Suppose that \(x\) and \(y\) are real numbers. Prove that if \(x + y\) is irrational, then \(x\) is irrational or \(y\) is irrational.

13. Let \(x\) be an irrational number. Prove that \(x^4\) or \(x^5\) is irrational.

14. Use a proof by contradiction to prove the following.

   There exist no natural numbers \(m\) such that \(m^2 + m + 3\) is divisible by 4.

15. Let \(a, b\) be distinct primes. Then \(\log_a(b)\) is irrational.

16. Prove or disprove the statement: there exists an integer \(n\) such that \(n^2 - 3 = 2n\).

17. Prove or disprove the statement: there exists a real number \(x\) such that \(x^4 + 2 = 2x^2\).

18. Prove that there exists a unique real number \(x\) such that \(x^3 + 2x = 2\).

19. Disprove that statement: There exists integers \(a\) and \(b\) such that \(a^2 + b^2 \equiv 3 \pmod{4}\).

20. Use induction to prove that 6\(|(n^3 + 5n)\) for all \(n \geq 0\).

21. Use induction to prove that \(1 \cdot 4 + 2 \cdot 7 + \cdots + n(3n + 1) = n(n + 1)^2\) for all \(n \in \mathbb{N}\).

22. Use the Strong Principle of Mathematical Induction to prove that for each integer \(n \geq 13\), there are nonnegative integers \(x\) and \(y\) such that \(n = 4x + 5y\).

23. A sequence \(\{a_n\}\) is defined recursively by \(a_0 = 1, a_1 = -2\) and for \(n \geq 1\),

   \[a_{n+1} = 5a_n - 6a_{n-1} \text{.}\]

   Prove that for \(n \geq 0\),

   \[a_n = 5 \times 2^n - 4 \times 3^n \text{.}\]

24. Suppose \(R\) is an equivalence relation on a set \(A\). Prove or disprove that \(R^{-1}\) is an equivalence relation on \(A\).

25. Consider the set \(A = \{a, b, c, d\}\), and suppose \(R\) is an equivalence relation on \(A\). If \(R\) contains the elements \((a, b)\) and \((b, d)\), what other elements must it contain?

26. Let \(A = \{a_1, a_2, a_3\}\) and \(B = \{b_1, b_2\}\). Find a relation on \(A \times B\) that is transitive and symmetric, but not reflexive.

27. Suppose \(A\) is a finite set and \(R\) is an equivalence relation on \(A\).
35. Consider the set $S$.

34. Prove the following statement: A nonempty set $A$ has the same cardinality as $B$ if and only if there exists an injective function $g : A \rightarrow B$. Is $g$ a bijective function?

33. Prove that the interval $(0, 1)$ is numerically equivalent to the interval $(0, +\infty)$.

32. (X points) Let $\mathbb{R}^+$ denote the set of positive real numbers and let $A$ and $B$ be denumerable subsets of $\mathbb{R}^+$. Define $C = \{x \in \mathbb{R} : -x/2 \in B\}$. Show that $A \cup C$ is denumerable.

31. Prove that $3$ is a prime number.

30. Suppose $f : A \rightarrow B$ and $g : X \rightarrow Y$ are bijective functions. Define a new function $h : A \times X \rightarrow B \times Y$ by $h(a, x) = (f(a), g(x))$. Prove that $h$ is bijective.

29. Consider the relation $S \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by $S = \{(x \mod 4, 2x \mod 6) \mid x \in \mathbb{Z}\}$.

28. Consider the relation $R \subset \mathbb{Z}_4 \times \mathbb{Z}_6$ defined by $R = \{(x \mod 4, 3x \mod 6) \mid x \in \mathbb{Z}\}$.

27. Prove that $R$ is a function from $\mathbb{Z}_4$ to $\mathbb{Z}_6$. Is $R$ a bijective function?

26. (a) Suppose $A$, $B$ are sets. Prove that if $A$ and $B$ have the same cardinality, then $A \times Z$ and $B \times Z$ have the same cardinality.

25. Prove that $R - S$ is uncountable.

24. Prove the following statement: A nonempty set $S$ is countable if and only if there exists an injective function $g : S \rightarrow \mathbb{N}$.

23. Consider the set $S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$. Prove that $R - S$ is uncountable.

22. (a) Suppose $A$, $B$ are sets. Prove that if $A$ and $B$ have the same cardinality, then $A \times Z$ and $B \times Z$ have the same cardinality.

21. Prove that $\mathbb{Z}^n$ has the same cardinality as $\mathbb{Z}^{n+1}$ for all $n \in \mathbb{N}$. Hint: Induct on $n$, and use part (a) for the inductive step.

20. (b) Prove that $\mathbb{Z}^n$ is countable for all $n \in \mathbb{N}$.

19. Compute the greatest common divisor of 42 and 13 and then express the greatest common divisor as a linear combination of 42 and 13.

18. Let $a, b, c \in \mathbb{Z}$. Prove that if $c$ is a common divisor of $a$ and $b$, then $c$ divides any linear combination of $a$ and $b$.

17. Define the term “$p$ is a prime”. Then prove that if $a, p \in \mathbb{Z}$, $p$ is prime, and $p$ does not divide $a$, then $\gcd(a, p) = 1$.

16. The greatest common divisor of three integers $a, b, c$ is the largest positive integer which divides all three. We denote this greatest common divisor by $\gcd(a, b, c)$. Assume that $a$ and $b$ are not both zero. Prove the following equation:

$$\gcd(a, b, c) = \gcd(\gcd(a, b), c).$$

15. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove the following:

$$\lim_{n \to \infty} \frac{3n + 1}{n - 2} = 3.$$

14. By using the formal definition of the limit of the sequence, without assuming any propositions about limits, prove that

$$\lim_{n \to \infty} \frac{(-1)^n 3n + 1}{n - 2}$$

does not exist.

13. Let $(a_n)$ be a sequence with positive terms such that $\lim_{n \to \infty} a_n = 1$. By using the formal definition of the limit of the sequence, prove the following:

$$\lim_{n \to \infty} \frac{3a_n + 1}{2} = 2.$$

12. (a) Use induction to prove

$$\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \cdots + \frac{1}{2n(2n + 2)} = \frac{n}{4(n + 1)}$$

for all $n \in \mathbb{N}$.

11. (b) Prove $\sum_{k=1}^{\infty} \frac{1}{2k(2k + 2)} = \frac{1}{4}$.