0. Ch3.1 A trivial proof and a vacuous proof (Reading assignment)
1. Ch3.2 Direct proofs
2. Ch3.3 Proof by contrapositive
3. Ch3.4 Proof by cases
4. Ch3.5 Proof evaluations (Reading assignment)

Ch 3.2: Direct proofs

A direct proof is a way of showing that a given statement is true or false by using existing lemmas and theorems without making any further assumptions. To prove statements of the form “if $P$, then $Q$”,

Assume that the statement $P$ is true and directly derive the conclusion that the statement $Q$ is true.

We can use the following properties of integers without justification.
- The sum (difference, product) of every two integers is an integer.
- The product of two negative integer is positive.
- Every integer is of the form $2m$ or $2m+1$, where $m \in \mathbb{Z}$.

Definition: An integer $x$ is called even (respectively odd) if there is an integer $k$ for which $x = 2k$ (respectively $2k+1$).

Example. If $n$ is an even integer, then $7n + 4$ is also an even integer.

Write a hypothesis and a conclusion first and fill out the body of the proof which is a bridge of logical deductions from the hypothesis to the conclusion.

Proof.
Exercises

1. Let $n \in \mathbb{Z}$. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

2. Let $S = \{0, 1, 2\}$ and let $n \in S$. Prove that if $(n + 1)^2(n + 2)^2/4$ is even, then $(n + 2)^2(n + 3)^2/4$ is even.

Proofs Involving Inequalities

(A1) For all real numbers $a, b, c$, if $a \leq b$ and $b \leq c$ then $a \leq c$.

(A2) For all real numbers $a, b, c$, if $a \leq b$ then $a + c \leq b + c$.

(A3) For all real numbers $a, b, c$, if $a \leq b$ and $0 \leq c$ then $ac \leq bc$.

Prove the statements below using A1-A3, together with any basic facts about $equality = .$

(1) For all real numbers $a, b$, if $0 \leq a$ and $a \leq b$ then $a^2 \leq b^2$. 
(2) For all real numbers $a$, if $a \leq 0$ then $0 \leq -a$.

(3) For all real numbers $a, b$, if $b \leq a$ and $a \leq 0$, then $a^2 \leq b^2$.

(4) For all real numbers $b$, $0 \leq b^2$.

(5) For all real numbers $a, b$, $ab \leq \frac{1}{2}(a^2 + b^2)$. *Hint: Consider $(a - b)^2$.*

(6) For all real numbers $a, b, \delta$, if $\delta \neq 0$ then $ab \leq \frac{1}{2}(\delta^2 a^2 + \delta^{-2}b^2)$.

(7) For all real numbers $a, b$, $ab = \frac{1}{2}(a^2 + b^2)$ if and only if $a = b$. 
Working Backwards

**Theorem** (Inequality between arithmetic and geometric mean.)

If \( a, b \in \mathbb{R} \) are such that \( a \geq 0 \) and \( b \geq 0 \), then \( \frac{a + b}{2} \geq \sqrt{ab} \)

**Scratch work:**
1. Start with the inequality you are asked to prove.
2. Simplify it as much as possible until you arrive at a statement that is obviously true.

**Formal Proof:**
3. In order to write formal proof, now start from the obviously true statement.
4. Use your previous work to guide you on how to arrive at the desired inequality.

**Proof**:

1. **What is wrong with this proof?**
   (1) Assume \( a = b \).
   (2) Multiplying both sides by \( b \), \( ab = b^2 \).
   (3) Subtracting \( a^2 \) from both sides, \( ab - a^2 = b^2 - a^2 \).
   (4) Factoring \( a(b - a) = (b + a)(b - a) \).
   (5) Dividing by \( b - a \), \( a = b + a \).
   (6) Using (1), \( a = 2a \).
   (7) Dividing by \( a \), \( 1 = 2 \).
2. **Circular argument** Prove if \( n^3 \) is even then \( n \) is even.

*Proof:*

Assume \( n^3 \) is even.
Then \( \exists k \in \mathbb{Z} \) such that \( n^3 = 8k^3 \).
It follows that \( n = (8k^3)^{1/3} = 2k \).
Therefore \( n \) is even.

All statements in the proof are true but is the proof correct?

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**Ch 3.3: Proof by contrapositive**

It is a direct proof but we start with the contrapositive because

\[
P \implies Q \text{ is equivalent to } \sim (Q) \implies \sim (P).
\]

**Why** do we prove the contrapositive of the implication instead of the original implication?

**Example.** Prove: If \( n^3 \) is even then \( n \) is even.
Definition: Two integers are said to have the same parity if they are both odd or both even.

Theorem: Let \( x, y \in \mathbb{Z} \). Then \( x \) and \( y \) are of the same parity if and only if \( x + y \) is even.

Proof.

Exercises

1. Prove that if \( 5x - 11 \) is an even integer, then \( x \) is an odd integer.

2. Prove that if \( 7x + 4 \) is even, then \( 5x - 11 \) is odd.
   (This problem is a continuation of the previous question. Of course we can prove this statement directly by using the definition of an even and odd integers)