Definition 1 (A relation): Let $X$ and $Y$ be sets.

A relation $R$ from $X$ to $Y$ is a subset of $X \times Y$.

Example 1. Let $A = \{1, 2\}$, $B = \{a, b, c\}$. List 3 relations from $A$ to $B$. How many possible relations are there from $A$ to $B$?

Definition 2. Let $R$ be a relation from $X$ to $Y$.

The domain of $R$ is

$$\text{dom}(R) = \{x \in X : (x, y) \in R \text{ for some } y \in Y\}.$$ 

The range of $R$ is

$$\text{range}(R) = \{y \in Y : (x, y) \in R \text{ for some } x \in X\}.$$ 

Notation: If $(x, y) \in R$, we write $xRy$.

Definition 3 (Inverse Relation): Let $R$ be a relation from $X$ to $Y$. The inverse relation of $R$ is

$$R^{-1} = \{(y, x) : (x, y) \in R\}.$$ 

Example 2. Find $R^{-1}$ for the relations you constructed above.

- What is the domain of $R^{-1}$?
- What is the range of $R^{-1}$?
Properties of Relations

Definition 4. A relation from a set $X$ to $X$ is reflexive if $xRx \forall x \in X$.

Give an example of a relation which is reflexive and a relation which is not reflexive.

Examples. Determine if the following relations are reflexive or not.

1. The relation $S$ is defined on $\mathbb{R}$ by $aSb$ if $a < b$.

2. The relation $\sim$ is defined on $\mathbb{R}$ by $x \sim y$ if $x \mid y$.

3. The relation $Q$ is defined on $\mathbb{R}$ by $(a, b) \in Q$ if $a \leq b$.

Definition 5. A relation from a set $X$ to $X$ is symmetric if whenever $xRy$, then $yRx$.

Give an example of a relation which is symmetric and a relation which is not symmetric.

Examples. Determine if the following relations are symmetric or not.

1. The relation $S$ is defined on $\mathbb{R}$ by $aSb$ if $a < b$.

2. The relation $\sim$ is defined on $\mathbb{R}$ by $x \sim y$ if $x \mid y$.

3. The relation $P$ is defined on $\mathbb{R}$ by $(a, b) \in P$ if $\frac{a}{b} \in \mathbb{Q}$.
Definition 6. A relation from a set $X$ to $X$ is transitive if whenever $xRy$ and $yRz$, then $xRz$.

Give an example of a relation which is transitive and a relation which is not transitive.

Examples. Determine if the following relations are transitive or not.

(1) The relation $S$ is defined on $\mathbb{R}$ by $aSb$ if $a < b$.

(2) The relation $\sim$ is defined on $\mathbb{R}$ by $x \sim y$ if $x \mid y$.

(3) The relation $M$ is defined on $\mathbb{R}$ by $(a, b) \in Q$ if $|a - b| < 1$.

Equivalence Relations

Definition 7 (An equivalence relation): A relation from a set $X$ to $X$ is an equivalence relation if it is reflexive, symmetric and transitive.

Definition 8. For an equivalence relation $R$ defined on a set $X$ and for $a \in X$, the set

$$[a] = \{ x \in X : xRa \}$$

consisting of all elements $x$ related to $a$ is called the equivalence class of $a$.

- Can $[a]$ be an empty set?

- Find the equivalence classes determined by

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$
Examples

(a) Are the following relations equivalence relations?

(b) If not, explain why not.

(c) If they are, prove that they are and

(d) determine the distinct equivalence classes.

1. The relation \(\simeq\) is defined on the set \(H = \{5k : k \in \mathbb{Z}\}\) by \(a \simeq b\) if \(a - b \in H\).

2. The relation \(P\) is defined on \(\mathbb{R} \times \mathbb{R}\) by \((a, b)R(c, d)\) if \(|a - c| \leq 3\).

3. The relation \(\sim\) is defined on \(\mathbb{R} \times \mathbb{R}\) by \((a, b) \sim (c, d)\) if \(a^2 - b = c^2 - d\).