Existence Proofs

Our goal in this section is to prove a statement of the form

\[ \text{There exists } x \text{ for which } P(x). \text{ (That is, } \exists x, P(x)). \]

I. A constructive proof of existence: The proof is to display a specific value \( x = a \) in a given set and verify that \( P(a) \) is true.

**EX:** Prove that, for every natural number \( x \), there exists a natural number \( y \) such that \( 2x - y = -1 \).

II. A nonconstructive proof of existence: Use theorems which imply the existence of an \( x \) such that \( P(x) \) is true without indicating how to explicitly produce such \( x \).

The *Intermediate Value Theorem* and the *Mean Value Theorem* are examples of existence theorems that can be used in this manner.

**EX:** Prove that there exists a real number \( x \) in \([-1, 1]\) such that \( 2x^3 + 1 = 0 \).

**Examples:** A constructive proof of existence

1. Prove that there are pairs of irrational numbers \( x \) and \( y \) such that \( x^y \) is rational.
2. Prove that there exists an integer \( x \) such that \( \frac{8x+2}{3x-1} = 2 \).

3. There exist distinct perfect squares \( x, y, \) and \( z \) such that \( x + y = z \).

4. Prove that for \( \varepsilon = 1 \), there exists a positive real number \( \delta \) such that
   \[ |x - 2| < \delta \implies |(2x + 3) - 7| < \varepsilon. \]
   (We will revisit the formal definition of a limit of a function in Chapter 12.)
5. There is a prime number $p$ such that $p + 2$ and $p + 6$ are also prime numbers.

6. There exists an even integer $n$ that can be written in two different ways as a sum of two distinct primes.

**Examples: A nonconstructive proof of existence**

- **The Intermediate Value Theorem**: If $f$ is a function that is continuous on the closed interval $[a, b]$ and $k$ is a number between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = k$.

- **The Mean Value Theorem**: If a function $f$ is continuous on the closed interval $[a, b]$, and differentiable on the open interval $(a, b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$.

**Examples**

1. There exists a solution for the equation $x^3 + 3x - 2 = 0$ in the interval $(0, 1)$. 

2. Let \( f(x) = 4x^5 - x + 2 \). Prove that there exists a \( c \in (0, 1) \) such that \( f'(c) = 3 \). Note that \( f(0) = 2 \) and \( f(1) = 5 \).

Unique Existence. Examples.

1. An equation \( x^5 + 4x - 1 = 0 \) has exactly one solution.

2. Prove that, for every \( x \), there exists a unique \( y \in \mathbb{R} \) such that \( 2x + 1 = 2y - 1 \).

   (1) Prove the existence \( y \) to \( 2x + 1 = 2y - 1 \).

   (2) Prove the uniqueness by contradiction.
Disproving Existence Statements

\[ \sim (\exists x \in S, P(x)) \equiv \forall x \in S, \sim P(x) \]

If the statement, “\(\exists x \in S, P(x)\)”, is false, every \(x \in S\) satisfies “\(\sim P(x)\)”. 

Examples: Disprove the statements

1. There is a real number \(x\) for which \(x^4 - 6x^2 + 2 < -7\).

2. There exist odd integers \(a\) and \(b\) such that \(4|(3a^2 + 7b^3)\). (Textbook).