All mathematical objects are formally defined in terms of sets and lists. Also, real-world objects usually have a natural mathematical model in these terms.

**Definition**: A *set* is any collection of objects (elements), either written out or specified by some condition. Order of elements is irrelevant, and no repeated elements are allowed. We enclose a set with curly brackets \{\}. 

**Examples**:

- \( A = \{1, 3, 5\} = \{3, 5, 1\} \).
- \( \mathbb{N} = \{\text{all natural numbers}\} = \{0, 1, 2, 3, \ldots\} \).
- \( \mathbb{R} = \{\text{all real numbers}\} \).
- \( B = \{x \in \mathbb{R} \text{ such that } x^2 = 2\} = \{\sqrt{2}, -\sqrt{2}\} \).
- \( C = \{x \in \mathbb{N} \text{ such that } x^2 = 2\} = \{\}, \) the empty set, since there is no natural-number solution to \( x^2 = 2 \).
- \( D = \{x \in \mathbb{R} \text{ such that } x^2 > 1\} = \{x \in \mathbb{R} \text{ such that } x > 1 \text{ or } x < -1\} \).

**Definition**: A *list* is an ordered sequence of any objects (entries), with repeat entries allowed. We enclose a list with round parentheses ( ).

**Examples**:

- \((1, 3, 5) \neq (1, 1, 3, 5) \neq (1, 5, 1, 3)\).
- An infinite sequence is a list, such as the even numbers \( a_n = 2n \):
  \[
  (a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \ldots) = (2, 4, 6, \ldots)
  \]

**Definition**: The *Cartesian product* \( A \times B \) is the set of all pairs \((a, b)\) with \( a \) drawn from \( A \) and \( b \) drawn from \( B \):

\[
A \times B = \{(a, b) \text{ for all } a \in A, \ b \in B\}.
\]

We use the times symbol \( \times \) because of the counting formula: \( |A \times B| = |A| \cdot |B| \).
Sample definitions

Coordinate geometry. A point in the coordinate plane is given by a list of two real-number coordinates, \( P = (x, y) \), such as the origin \((0, 0)\). The plane is just the set of all such points:

\[
\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \text{ for all } x, y \in \mathbb{R}\}.
\]

The line \( L \) through the points \((1, 0)\) and \((0, 1)\) is the set of all points (infinitely many) satisfying a certain linear equation:

\[
L = \{(x, y) \in \mathbb{R}^2 \text{ s.t. } x + y = 1\}.
\]

The points \((1, 0), \left(\frac{1}{3}, \frac{2}{3}\right), (-\sqrt{2}, \sqrt{2}-1)\) are some elements of \( L \).

Playing cards. Cards in a standard deck are distinguished by two pieces of data: a face value \( A, 2, 3, \ldots, 10, J, Q, \text{ or } K; \) and a suit \( \spadesuit, \heartsuit, \diamondsuit, \text{ or } \clubsuit \). It is natural to model a card as a pair list: e.g. the queen of spades corresponds to \( c = (Q, \spadesuit) \). Then the deck is the cartesian product \( D = F \times S \), where:

\[
F = \{A, 2, 3, \ldots, 10, J, Q, K\} \quad \text{and} \quad S = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}.
\]

Note that \( D \) is naturally a set rather than a list, because the deck remains the same regardless of how it is shuffled.

A hand \( H \) is an unordered collection of cards from the deck: that is, a subset \( H \subset D \). To study poker odds, we put all possible 5-card hands into a set:

\[
P = \{H \subset D \text{ s.t. } |H| = 5\}.
\]

Notice that each element \( H \in P \) is itself a set, and each element \( c \in H \) is itself a list (a pair). All kinds of real-world objects can be modeled by some such layer-cake of sets and lists.

Ordered pair definition of a function. A function \( f : A \to B \) is any “rule” taking each input \( a \in A \) to an output \( b = f(a) \in B \). But if two rules give identical outputs for each input, then they define the same function. To formalize this, we give a function by its “table of values”:

\[
f = \{(a, b) \in A \times B \text{ s.t. } b = f(a)\} = \{(a, f(a)) \text{ for all } a \in A\}.
\]
For example, the function \( f(n) = n^2 + n \) on natural numbers \( n \in \mathbb{N} \) is defined by the table:

\[
\begin{array}{c|c|c|c|c}
\hline
n & 0 & 1 & 2 & 3 & \cdots \\
\hline
f(n) & 0 & 2 & 5 & 12 & \cdots \\
\hline
\end{array}
\]

which can be thought of as the set of all pairs of inputs and outputs:

\[
f = \{(0, 0), (1, 2), (2, 5), (3, 12), \ldots \} \\
= \{(n, n^2+n) \text{ for all } n \in \mathbb{N}\}.
\]

Note that the function \( g(n) = n(n + 1) \) is defined by a different formula, but gives the same values, the same set of pairs, and hence the same function: \( f = g \).

**What is formal mathematics?** The above set-and-list definitions seem to forget the “real meaning” of the objects they model. For example, a card is a marked piece of paper, not a pair like \((Q, \spadesuit)\). Yet this data identifies each card, so the formal definition is just what we need to model and solve any problem about card-hands. Similarly, the plane coordinates \((x, y)\) don’t describe what a geometric point is, but they determine where it is, and this is all that is relevant to any geometric problem.

For a function on the real numbers, the set-of-pairs definition gives the same formal object as a set of points in the plane: for example, the function \( \ell(x) = -x + 1 \) for \( x \in \mathbb{R} \) gives:

\[
\ell = \{(x, y) \in \mathbb{R}^2 \text{ s.t. } y = -x + 1\},
\]

which is just the line \( L \) in a previous example. Why this coincidence? Because the line is the graph of the function! The function and its graph are defined by the same data, so formally, the function is its graph.

Although formal definitions in terms of sets and lists do not describe the “real meaning” of the objects, they contain all the identifying data. For formal reasoning, we work only with the set-theory data, and leave the intuitive meaning for informal discussion.