In Supplement 9/9, we defined the choose number, or binomial coefficient, \(^n_k\) to be the number of possible \(k\)-element subsets \(S \subseteq [n]\), where \([n] = \{1, 2, \ldots, n\}\). For example, \(^4_2 = 6\) counts the \(2\)-element subsets \(S \subseteq \{1, 2, 3, 4\}\), namely: \(S = \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\).

We can put these numbers into an array called Pascal’s Triangle (in China, Yang Hui’s Triangle; in Persia, Khayyam’s Triangle):

\[
\begin{array}{ccccccc}
0 & & & & & & 1 \\
1 & & & & & 1 & \\
0 & 1 & & & & 1 & \\
1 & 2 & 1 & & 1 & \\
1 & 3 & 3 & 1 & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

We can compute the entries by the formula \(^n_k\) = \(\binom{n-1}{k-1} + \binom{n-1}{k}\), but there is an easier way. It is a remarkable fact that each entry in the triangle is the sum of the two entries immediately above it (except for the edges \(^n_0\) = \(^n_n\) = 1). For example, the next row will be:

\[
\begin{array}{ccccccc}
\binom{0}{0} & 1 & 1 & & & & \\
\binom{1}{0} & \binom{1}{1} & 1 & 1 & & & \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} & 1 & 2 & 1 & \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & 1 & 3 & 3 & 1 \\
\binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & 1 & 4 & 6 & 4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

In general, the recurrence formula is:

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

**Problem 1.** Use the above recurrence to compute the \(^6_0\) and \(^7_3\) rows of the table.

**Problem 2.** Find the sum of each row: \(^0_0\) + \(^1_1\) + \cdots + \(^n_n\), for \(n = 0, 1, \ldots, 7\). Guess a general formula for this sum.

**Problem 3.** Prove your formula from Prob. 2 using a proposition from Supplement 9/9.

We can prove the recurrence formula \((*)\) through the Bijection Principle:

- \(^n_k\) = \(|A|\), where \(A\) is the set of all \(k\)-element subsets of \([n]\).
- \(^{n-1}_{k-1}\) = \(|B_1|\), where \(B_1\) is the set of all \((k-1)\)-element subsets of \([n-1]\).
- \(^{n-1}_k\) = \(|B_2|\), where \(B_2\) is the set of all \(k\)-element subsets of \([n-1]\).

If we can give a bijection \(\phi: A \rightarrow B_1 \cup B_2\), then this will show that \(|A| = |B_1| + |B_2|\), which is precisely the recurrence formula \(^n_k\) = \(^{n-1}_{k-1}\) + \(^{n-1}_k\).

The bijection mapping \(\phi\) is defined on \(k\)-element subsets of \([n]\) by: \(\phi(S) = S' = S \setminus \{n\}\), meaning we remove \(n\) from \(S\) if it is present, and leave \(S' = S\) otherwise. The result \(S'\) is a subset of \([n-1]\) with either \(k-1\) or \(k\) elements.

For example, the bijection \(\phi\) for \(^3_2 = \binom{3}{1} + \binom{3}{2}\) is given in the table, where \(S' = \phi(S)\):

<table>
<thead>
<tr>
<th>(S \in A)</th>
<th>{1,2}</th>
<th>{1,3}</th>
<th>{1,4}</th>
<th>{2,3}</th>
<th>{2,4}</th>
<th>{3,4}</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S' \in B_1)</td>
<td>{1,2}</td>
<td>{1,3}</td>
<td>{1,4}</td>
<td>{2,3}</td>
<td>{2,4}</td>
<td>{3,4}</td>
</tr>
<tr>
<td>(\in B_2)</td>
<td>{1}</td>
<td>{1,3}</td>
<td>{2,3}</td>
<td>{2}</td>
<td>{3}</td>
<td></td>
</tr>
</tbody>
</table>
Problem 4. Illustrate the mapping φ in the case of \( \binom{5}{3} = \binom{4}{2} + \binom{4}{3} \). Make a table like the one above.

Problem 5. Formally define the inverse mapping ψ: \( B_1 \cup B_2 \rightarrow A \), which undoes φ. That is, given a subset \( S' \subset [n-1] \) with either \( k-1 \) or \( k \) elements, define the corresponding \( k \)-element \( S \subset [n] \).

Take a fairly large example of \( S \), and verify that \( ψ(φ(S)) = S \); also take an example of \( S' \) and verify that and \( φ(ψ(S')) = S' \).