GENERAL HELICES

AND

OTHER TOPICS IN THE DIFFERENTIAL GEOMETRY OF CURVES

By

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ABSTRACT

The purpose of this report is three-fold. First, a possible motivation for the allowability conditions of a curve is presented; whereas, the usual approach is often to start from the allowability conditions. Secondly, the basic results of the differential geometry of curves are summarized and organized; the results having the same assumptions are collected together in one chapter, and the chapters are organized so that the material in any one chapter will hold in subsequent chapters. This differs from the usual approach in that the results are not often grouped together by their hypotheses. Finally, and most importantly, a variety of important properties and results involving general helices is presented. General helices prove to be a important class of curves, because they possess a very simple property: "the ratio of their torsion to curvature is constant." This property greatly simplifies a lot of the more complicated formulas in differential geometry. Additionally, recent work by begun by Konrad Heuvers demonstrates that an association between general helices and plane curves exists. This association allows for the possibility of studying general helices by studying associated plane curves, which is a much simpler task.

Contents

Notation

1. $\underline{f}$: an underscore underneath a letter, indicates that the letter represents a function
2. $\overrightarrow{\underline{f}}$: represents a vector-valued function
3. $|\overrightarrow{v}|$: magnitude of vector $\overrightarrow{v}$
4. $\langle x, y, z\rangle$: shorthand for $x\widehat{i} + y\widehat{j} + z\widehat{k}$
5. $C^{(r)}$: class r; a curve of class r has the components of its allowable representation to be r times continuously differentiable s: usually used to indicate arc length parameter
6. $[\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}]$: the *box product* or *triple scalar product* of three vectors $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$. This is $\overrightarrow{u} \cdot (\overrightarrow{v} \times \overrightarrow{w})$, or equivalently, $(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{w}$

Chapter 1

Preliminaries

1.1 Introduction

What is a curve? To answer this question, we consider the properties that we wish a curve to have, and then use these to guide our search for an appropriate definition. Intuitively, we think of a curve as some "winding path" through space. If we imagine each point along the path as having a label, then identifying a curve would amount to making a correspondence between the labels and the actual coordinates in space assigned to the point. For this purpose, we consider "mappings".

1.2 Mappings

Definition 1.2.1 Let A and B be sets. A rule $\underline{\mathrm{f}}$ which associates every point of A to exactly one corresponding point of B is called a *function* or a *mapping* of A *into* B. We symbolize this by

$$\underline{\mathrm{f}} : A \to B.$$

In this situation, A is called the *domain* of $\underline{\mathrm{f}}$ and B is called the *codomain* of $\underline{\mathrm{f}}$. If $\underline{\mathrm{f}}$ maps $a \in A$ to $b \in B$, we write $\underline{\mathrm{f}}(a) = b$. We refer to b as the *image* of a under $\underline{\mathrm{f}}$ and a as the *preimage* of b.

Remark An underscore is used to identify a function, i.e. $\underline{\mathrm{f}}$. This proves useful in certain situations, where it may be desirable to give the output variable the same letter name as the function.

Definition 1.2.2 A mapping $\underline{\mathrm{f}}$ is said to map A *onto* B if for every $b \in B$ there exists an $a \in A$ such that $\underline{\mathrm{f}}(a) = b$. A mapping $\underline{\mathrm{f}}$ is said to be *one-to-one* if for any $a_1, a_2 \in A$ with $a_1 \neq a_2$, $\underline{\mathrm{f}}$ maps a_1 and a_2 to different points in B, i.e. $\underline{\mathrm{f}}(a_1) \neq \underline{\mathrm{f}}(a_2)$.

Remark If $\underline{\mathrm{f}}$ is a one-to-one mapping of A onto B, then $\underline{\mathrm{f}}$ is called a *bijection*, and an *inverse* mapping $\underline{\mathrm{f}}^{-1}$ exists that maps B in a one-to-one fashion onto A.

1.3 Bijective Maps

In attempting to define an arc of a curve, we could let the set of point "labels" be the set of numbers in an interval $[a, b]$, where $a, b \in \mathbb{R}$. We could then try to describe the arc by defining a mapping from $[a, b]$ to the arc's spatial points. Since we desire each number in $[a, b]$ to correspond to exactly one arc point, and each arc point to correspond to exactly one number in $[a, b]$, we naturally look to bijective maps.

However, for our purposes, this proves to be inadequate. We do not want to consider a two-dimensional region to be the arc of a curve, but bijective maps alone do not discount this possibility.

Theorem 1.3.1 *The interval* $[0, 1]$ *can be mapped in a one-to-one manner onto the square* $[0, 1] \times [0, 1]$.

Proof: Let $t \in [0, 1]$. Write t uniquely as an infinite decimal, using repeating 9's if necessary. Hence, if t is rational, say $t = \frac{1}{8} = .125$, write t as $.124999\ldots$. For any t, we have $t = .c_1c_2\ldots c_n\ldots$. Define c_{n_1} to be the first nonzero digit in the expansion of t, and moreover, for all $k \geq 1$, define c_{n_k} to be the kth nonzero digit in the expansion of t. Hence we can write $t = .c_1\ldots c_{n_1}c_{n_1+1}\ldots c_{n_2}c_{n_2+1}\ldots c_{n_3}c_{n_3+1}\ldots c_{n_4}c_{n_4+1}\ldots$ Now define $\underline{\text{f}} : [0, 1] \to [0, 1] \times [0, 1]$ by letting

$$\underline{\text{f}}(t) = (.c_1\ldots c_{n_1}c_{n_2+1}\ldots c_{n_3}c_{n_4+1}\ldots\ ,\ .c_{n_1+1}\ldots c_{n_2}c_{n_3+1}\ldots c_{n_4}c_{n_5+1}\ldots).$$

Since t is written uniquely, the mapping is well-defined. We now show that the mapping is, in fact, bijective. We can do this by showing that any point $(x_1, x_2) \in [0, 1] \times [0, 1]$ has exactly one inverse image.

Let $(x_1, x_2) \in [0, 1] \times [0, 1]$. Write each coordinate x_1 and x_2 as unique infinite decimals, using repeating 9's if necessary. Hence we have

$$x_1 = .d_1d_2\ldots d_n\ldots \text{ and } x_2 = .e_1e_2\ldots e_n\ldots.$$

Now for all $k \geq 1$, define d_{n_k} and e_{n_k} to be the kth nonzero digits in the expansions of x_1 and x_2 respectively. Therefore we can write

$$x_1 = .d_1\ldots d_{n_1}d_{n_1+1}\ldots d_{n_2}d_{n_2+1}\ldots d_{n_3}d_{n_3+1}\ldots d_{n_4}d_{n_4+1}\ldots.$$

$$x_2 = .e_1\ldots e_{n_1}e_{n_1+1}\ldots e_{n_2}e_{n_2+1}\ldots e_{n_3}e_{n_3+1}\ldots e_{n_4}e_{n_4+1}\ldots.$$

We then let $t = .d_1\ldots d_{n_1}e_1\ldots e_{n_1}d_{n_1+1}\ldots d_{n_2}e_{n_1+1}\ldots e_{n_2}d_{n_2+1}\ldots.$

Since x_1 and x_2 are uniquely represented and are infinite decimals, this inverse map produces a unique infinite decimal. Thus the inverse map is well-defined. Furthermore, the sequence of nonzero digits

$\langle c_{n_k}\rangle_{k=1}^{\infty}$ is precisely $< d_{n_1}, e_{n_1}, d_{n_2}, e_{n_2}\cdots >$ so we have $\underline{\text{f}}(t) = (x_1, x_2)$. ∎

1.4 Continuous Maps

Since bijective maps prove to be inadequate, we look to continuity.

Definition 1.4.1 A mapping $\underline{f} : A \to B$ is said to be *continuous at a point* $a \in A$ if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $|x - a| < \delta$, then $|\underline{f}(x) - \underline{f}(a)| < \varepsilon$. $\underline{f}$ is said to be *continuous* if it is continuous at every point in A.

Definition 1.4.2 The image of a continuous mapping is called a *Peano curve.*

The next section investigates the construction of a special Peano curve. In particular, we will discover that we encounter the same problem as before: we can map an interval to an entire two-dimensional region!

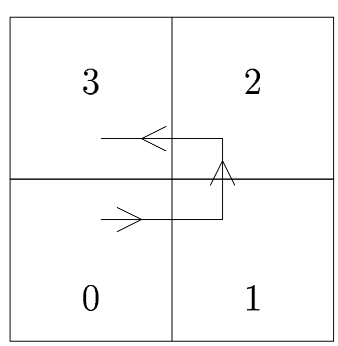
1.5 A Special Peano Curve

Before giving the explicit description of the Peano curve, we first describe a constructive, subdividing process of the unit square, which we will then use to construct the mapping. The process is as follows:

1. Let Q be the square $[0, 1] \times [0, 1]$. Subdivide Q into 4 squares q, labeling them with the boundaries as $0, 1, 2, 3$.
2. Subdivide each of these squares into four subsquares qr, where with the boundaries they are labeled $q0, q1, q2, q3$.
3. Repeat the subdivision process indefinitely.
4. Number the squares so that all of the squares are passed through, if you pass from one to the next in increasing order, whereby a polygon is generated having no multiple points.

Lemma 1.5.1 *The numbering process described in the construction is always possible.*

Proof: Let $\underline{P}(n)$ be the proposition that such a numbering is possible at the nth subdivision. $\underline{P}(1)$ is true by the following diagram.



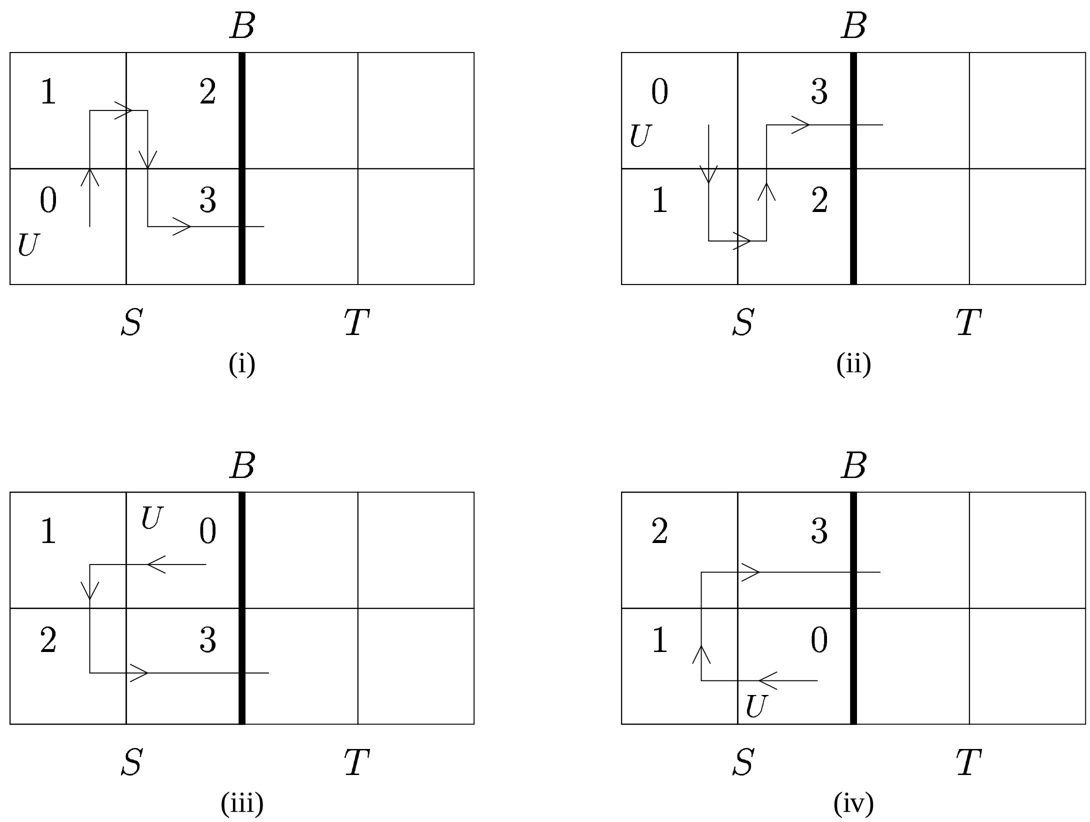


Figure 1.1: Diagrams used in the proof of Lemma 1.5.1

Assume that $\underline{\text{P}}(n)$ is true, so that the ordering is possible at the nth subdivision. Let S and T be any two consecutive squares, so that S is $q_1q_2\ldots q_n$ and T is $q_1q_2\ldots(q_n+1)$. Consider now the $(n+1)$th subdivision. Let B be the boundary between S and T. Without loss of generality, let us assume that B is on the right edge of S. From a possible square R that precedes S, one subsquare of S may be forced in advance to be $q_1q_2\ldots q_n0$, and let us call this U for convenience. No matter where U is, we have a polygonal path from U through the subsquares of S, finally crossing B into T. Since U could be in one of four locations, we identify all four possibilities by the diagrams in Figure 1.1. (Note: the numbers x in the subsquares of S represent the squares $q_1q_2\ldots q_nx$.)

Hence the ordering is possible at the $(n+1)$th subdivision, and $\underline{\text{P}}(n+1)$ is true. This shows that such a numbering process is always possible. ▮

Note that in this construction, once the lowest/first number in a new subsquare is identified, then the subsquare that is assigned the highest number will be adjacent to it and sitting next to the boundary of the next square.

We now introduce some useful lemmas that will allow us to assemble the preceding subdivision process and use it to formulate the Peano curve mapping.

Lemma 1.5.2 *Any point P of the original square Q is a point of every subsquare in at least one sequence* $q_1, q_1q_2, \ldots, q_1q_2\cdots$.

Proof: Let $P \in Q$. Let $\underline{\mathrm{P}}(n)$ be the proposition that P is in some subsquare $q_1 \ldots q_n$. We show that a sequence must exist, by demonstrating that for all n, a term of the sequence exists, and we do this by induction on n.

Since $P \in Q$, P is in at least one subsquare $0, 1, 2,$ or 3. Call one such subsquare q_1, so that $P \in q_1$, and $\underline{\mathrm{P}}(1)$ is true. Now assume that $\underline{\mathrm{P}}(n)$ is true, so that $P \in q_1 \ldots q_n$. Hence, P must be in at least one of $q_1 \ldots q_n0, q_1 \ldots q_n1, q_1 \ldots q_n2,$ or $q_1 \ldots q_n3$. Call one such subsquare $q_1 \ldots q_nq_{n+1}$. Therefore, $P \in q_1 \ldots q_nq_{n+1}$ and thus $\underline{\mathrm{P}}(n+1)$ is true. Hence, P is a point of every subsquare in the sequence $q_1, q_1q_2, \ldots, q_1q_2\ldots$. ∎

Lemma 1.5.3 *If $q_1, q_1q_2, \ldots, q_1q_2\ldots$ is a sequence of subsquares of Q, then the sequence converges to a single point in Q.*

Proof: Let $q_1, q_1q_2, \ldots, q_1q_2\ldots$ be a sequence of subsquares of Q. For all n, let subsquare $q_1 \ldots q_n$ be represented by $K_n \times L_n$. We note that there exist $a_n, b_n, c_n, d_n \in \mathbb{R}$ with $a_n \leq b_n$ and $c_n \leq d_n$, so that $K_n = [a_n, b_n]$ and $L_n = [c_n, d_n]$. Now $K_{n+1} = [a_{n+1}, b_{n+1}]$; by our construction process this will be either $[a_n, a_n + \frac{b_n - a_n}{2}]$ or $[a_n + \frac{b_n - a_n}{2}, b_n]$. Hence a_{n+1} is either a_n or $a_n + \frac{b_n - a_n}{2}$. Thus $a_{n+1} \geq a_n$. Therefore $\langle a_n \rangle_{n=1}^{\infty}$ is nondecreasing. Since $\langle a_n \rangle_{n=1}^{\infty}$ is bounded above by 1, $\langle a_n \rangle_{n=1}^{\infty}$ converges to some point $a \in [0,1]$. Similarly, b_{n+1} is either b_n or $b_n - \frac{b_n - a_n}{2}$ (same as $a_n + \frac{b_n - a_n}{2}$), so $b_{n+1} \leq b_n$. Therefore $\langle b_n \rangle_{n=1}^{\infty}$ is nonincreasing. Since $\langle b_n \rangle_{n=1}^{\infty}$ is bounded below by 0, $\langle b_n \rangle_{n=1}^{\infty}$ converges to some point $b \in [0,1]$. Furthermore, since $a_n \leq b_n$, for all n, we have $a \leq b$. So letting $K = [a, b]$, we have K_n converging to K. By the same argument, we have L_n converging to L, where $L = [c, d]$ for some $c, d \in [0,1]$ with $c \leq d$. Hence $K_n \times L_n$ converges to $K \times L$. However, by construction, $d_n - c_n = b_n - a_n = \frac{1}{2^n}$, so taking limits as $n \to \infty$, we have $d - c = b - a = 0$. Thus $a = b$ and $c = d$. Therefore, the sequence of subsquares converges to a single point in Q. ∎

Lemma 1.5.4 *For any point $P \in Q$, there exists a sequence of subsquares converging to it.*

Proof: Let $P \in Q$. By Lemma 1.5.2, there exists a sequence of subsquares $q_1, q_1q_2, \ldots, q_1q_2\ldots$, so that P is in every subsquare. By Lemma 1.5.3, this sequence must converge to a single point in Q, say P^*, so P^* must also be in every subsquare. Now for all n, both P and P^* lie in subsquare $q_1 \ldots q_n$, so the distance between them must be less than or equal to $\frac{\sqrt{2}}{2^n}$ for all n. Hence, the distance between P and P^* must be zero. This forces $P = P^*$, so there exists a sequence of subsquares converging to P. ∎

Lemma 1.5.5 *For all $t \in [0,1]$, and for all i, there exist $a_i \in \{0,1,2,3\}$, so that*

$$t = \sum_{i=1}^{\infty} \frac{a_i}{4^i}.$$

Either this representation is unique, or t has exactly two such representations, where

$$t = \sum_{i=1}^{\infty} \frac{a_i}{4^i} = \sum_{i=1}^{\infty} \frac{b_i}{4^i}$$

for $a_i, b_i \in \{0,1,2,3\}$. If t has two such representations, then there exists $n \geq 1$, so that $a_i = b_i$ for $i < n$, $a_n + 1 = b_n$, and $a_i = 3, b_i = 0$ for $i > n$ (or $b_n + 1 = a_n$ with $a_i = 0, b_i = 3$ for $i > n$).

Proof: By Ross [7, Theorem 16.3], this lemma is proved using representations $\sum_{i=1}^{\infty} \frac{c_i}{10^i}$ for $c_i \in \{0,1,2,3,4,5,6,7,8,9\}$. Only a slight modification of this proof is needed to justify the present lemma. ∎

Theorem 1.5.1 *The interval $[0,1]$ can be mapped continuously onto the square $[0,1] \times [0,1]$.*

Proof: Let $t \in [0,1]$. By Lemma 1.5.5, for all i, there exist $q_i \in \{0,1,2,3\}$, so that

$$t = \sum_{i=1}^{\infty} \frac{q_i}{4^i}.$$

Now define $\underline{f} : [0,1] \to [0,1] \times [0,1]$ by letting $\underline{f}(t)$ be the single point in $[0,1] \times [0,1]$ to which the sequence of subsquares $q_1, q_1 q_2, \ldots, q_1 q_2 \ldots$ converges by Lemma 1.5.3.

First we show that this mapping is well-defined. If the representation

$$t = \sum_{i=1}^{\infty} \frac{q_i}{4^i}$$

is unique, then $\underline{f}(t)$ corresponds to a single point. If not, then by Lemma 1.5.5, t has exactly two representations $\sum_{i=1}^{\infty} \frac{a_i}{4^i}$ and $\sum_{i=1}^{\infty} \frac{b_i}{4^i}$ where, without loss of generality, for some $n \geq 1$, $a_i = b_i$ for $i < n$, $a_n + 1 = b_n$, and $a_i = 3, b_i = 0$ for $i > n$. Let

$$\underline{f}\left(\sum_{i=1}^{\infty} \frac{a_i}{4^i}\right) = P_1$$

and

$$\underline{f}\left(\sum_{i=1}^{\infty} \frac{b_i}{4^i}\right) = P_2.$$

For all $r > n$, subsquare

$$a_1 \ldots a_n \ldots a_r = a_1 \ldots a_n 3 \ldots 3$$

is adjacent to subsquare

$$b_1 \ldots b_n \ldots b_r = a_1 \ldots a_{n-1}(a_n + 1) 0 \ldots 0$$

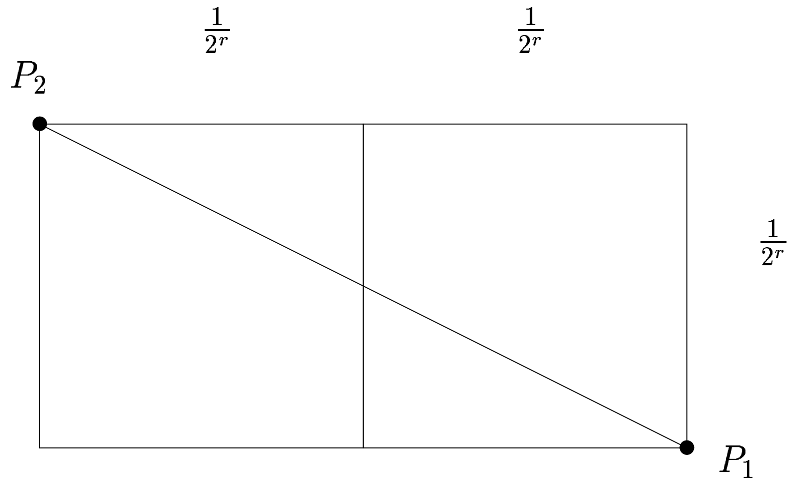


Figure 1.2: Diagram of maximum distance between P_1 and P_2

by our numbering process. So for P_1 and P_2, which lie in $a_1 \ldots a_n \ldots a_r$ and $b_1 \ldots b_n \ldots b_r$ respectively, the maximum distance between P_1 and P_2 is the length of the diagonal of a rectangle having sides $\frac{1}{2^r}$ and $\frac{2}{2^r}$, as can be seen by Diagram 1.2.

Hence, for all $r > n$, the maximum distance between P_1 and P_2 is

$$\sqrt{\left(\frac{2}{2^r}\right)^2 + \left(\frac{1}{2^r}\right)^2} = \frac{\sqrt{5}}{2^r}.$$

In other words, for all $r > n$,

$$d \leq \frac{\sqrt{5}}{2^r}.$$

where d is the maximum distance between P_1 and P_2. Therefore,

$$d \leq \lim_{r\to\infty} \frac{\sqrt{5}}{2^r} = 0.$$

So $P_1 = P_2$, which shows that two different representations of t map to the same point. Thus the mapping is well-defined.

Secondly, we show that this mapping is onto. Let P be any point in Q. By Lemma 1.5.4, there exists a sequence of subsquares converging to it, say $q_1, q_1q_2, \ldots, q_1q_2 \ldots$. Let

$$t = \sum_{i=1}^{\infty} \frac{q_i}{4^i}.$$

Now

$$0 \leq \sum_{i=1}^{\infty} \frac{q_i}{4^i} \leq \sum_{i=1}^{\infty} \frac{3}{4^i} = \frac{3}{4} \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i = \frac{3}{4}\left[\frac{1}{1-\frac{1}{4}}\right] = 1.$$

So $t \in [0,1]$, and $\underline{\text{f}}(t) = P$. Thus the mapping is onto.

Finally, we show that the mapping is continuous. To do this, we must show that for every $\varepsilon > 0$ and for every $t_1 \in [0,1]$, there exists a $\delta > 0$, such that if $|t_2 - t_1| < \delta$, then $\underline{\text{d}}(P_2, P_1) < \varepsilon$, where $P_1 = \underline{\text{f}}(t_1)$, $P_2 = \underline{\text{f}}(t_2)$, and $\underline{\text{d}}(P_2, P_1)$ represents the distance between P_2 and P_1. Now let $\varepsilon > 0$ and $t_1 \in [0,1]$.

Case I: $\varepsilon > \sqrt{2}$

Since P_1 and P_2 lie in the unit square, for every $\delta > 0$, we have $\underline{\mathrm{d}}(P_2, P_1) \leq \sqrt{2} < \varepsilon$.

Case II: $\varepsilon \leq \sqrt{2}$

There exists an $N_0 \geq 1$, so that $\sqrt{2}\left(\frac{1}{2}\right)^{N_0} \leq \sqrt{2}\left(\frac{1}{2}\right)^{N_0-1}$. Choose $\delta = \frac{1}{4^{N_0+1}}$. Let $|t_2 - t_1| < \delta$. Since $\underline{\mathrm{f}}$ is well-defined, we may choose representations for t_1 and t_2 so that the corresponding q_i's do not end with repeating 3's. Therefore $|t_2 - t_1| < \frac{1}{4^{N_0+1}}$ implies that P_1 and P_2 both lie in the subsquare $q_1 q_2 \ldots q_{N_0}$. Since this subsquare has edge length $\left(\frac{1}{2}\right)^{N_0}$, $\underline{\mathrm{d}}(P_2, P_1) \leq \sqrt{2}\left(\frac{1}{2}\right)^{N_0} \leq \varepsilon$.

Thus the mapping is continuous. ▮

Remark The mapping described in Theorem 1.5.1 is *not* one-to-one. A point on the boundary of one of the squares has several inverse images. Hence, while the mapping is continuous, it is not bijective.

1.6 Topological Maps

Both bijective maps and continuous maps are, in and by themselves, insufficient to use in the definition of an arc of a curve, since both allow for the possibility that an interval can be mapped onto an entire two-dimensional region.

We will now consider mappings that are both bijective *and* continuous.

Definition 1.6.1 A mapping $\underline{\mathrm{f}} : A \to B$ is said to be *topological* if $\underline{\mathrm{f}}$ is a bijective, continuous mapping, whose inverse mapping is continuous.

In our case, all we need to do is accept both bijectivity and continuity to consider topological mappings, by the following theorem.

Theorem 1.6.1 *If $\underline{\mathrm{f}} : [a, b] \to M$ is a bijective, continuous map, then $\underline{\mathrm{f}}$ is topological.*

Proof: Let $\underline{\mathrm{f}} : [a, b] \to M$ be a bijective, continuous map. Then, by Gaughan [3, Theorem 3.12], $\underline{\mathrm{f}}^{-1}$ is continuous. Hence, by definition, $\underline{\mathrm{f}}$ is topological. ▮

Definition 1.6.2 The topological image of an interval $[a, b]$ is called a *Jordan arc*.

The following theorem will demonstrate that topological mappings are the types of mappings for which we were looking.

Theorem 1.6.2 *A topological mapping preserves dimension. In particular, it is impossible for an interval to be topologically mapped to a two-dimensional region.*

Proof: This theorem was proved by Brouwer in 1911, [1]. ▮

Given Theorem 1.6.2, it seems as if the definition of a Jordan arc is the right one to use for the definition of the arc of a curve. However, since we desire to study curves using calculus, we tighten the definition further by adding differentiability. We will choose our assumptions about derivatives in such a way as to permit calculation, while ensuring that our mapping is topological.

Chapter 2

Curve Concepts

2.1 Vector Approach to Curve Representation

To consider the definition for the arc of a curve, we consider mappings $\underline{\mathrm{f}} : [a,b] \rightarrow M$, where M is the point set of the curve. Since we consider curves in 3-space, M will consist of ordered triples in Cartesian coordinates. Hence $\underline{\mathrm{f}}(t) = P$, where $P = (x_1, x_2, x_3)$. Thus

$$\begin{cases} x_1 = \underline{\mathrm{f}}_1(t) \\ x_2 = \underline{\mathrm{f}}_2(t) \\ x_3 = \underline{\mathrm{f}}_3(t) \end{cases} .$$

We can combine these three parametric equations into one vector equation as

$$\overrightarrow{x} = x_1\widehat{i} + x_2\widehat{j} + x_3\widehat{k} = \underline{\mathrm{f}}_1(t)\widehat{i} + \underline{\mathrm{f}}_2(t)\widehat{j} + \underline{\mathrm{f}}_3(t)\widehat{k} = \overrightarrow{\underline{\mathrm{f}}}(t).$$

Hence the output $\overrightarrow{\underline{\mathrm{f}}}(t)$ is a vector, but we consider the associated point on the curve to be the point at the head of the position vector of $\overrightarrow{\underline{\mathrm{f}}}(t)$. This vector approach has two obvious effects. For one, it streamlines the number of derived equations from 3 to 1. Secondly, it will make many theorems about curves independent of their locations in space, which is desirable since the geometry is not affected by location.

2.2 Allowable Representations

We will restrict the class of representations $\overrightarrow{\underline{\mathrm{f}}}(t)$ to those we consider to be *allowable*. The allowability conditions that we present here and in the next section follow those presented by Kreyszig [5].

An *allowable representation* $\overrightarrow{\underline{\mathrm{f}}}(t)$ is one in which the following conditions are met:

C1 $\overrightarrow{\underline{\mathrm{f}}}$ is defined on a finite interval $I = [a, b]$.

C2 Given $\vec{\underline{f}}(t) = \underline{f}_1(t)\hat{i} + \underline{f}_2(t)\hat{j} + \underline{f}_3(t)\hat{k}$, for all $i \in \{1,2,3\}$, the $\underline{f}_i$ are $r \geq 1$ times continuously differentiable in I.

C3 For every $t \in I$, there exists a $k \in \{1,2,3\}$, so that $\underline{f}'_k(t)$ is nonzero.

We will shortly show that the allowability conditions cause the output $\vec{\underline{f}}(t)$ to effectively be a special type of Jordan arc, but first we consider parametric transformations.

2.3 Parametric Transformations

We will consider the possibility of using different parametric representations as the need arises, so we consider the transformations between parameters and the conditions needed so that the transformations are allowable. The conditions we will choose are identical to those chosen for the allowability of $\vec{\underline{f}}(t)$. An *allowable parametric transformation*, $t = \underline{\nu}(t^*)$, is one in which the following conditions hold:

PT1 $\underline{\nu}$ is at least defined on an interval I^*, where $I^* = [a^*, b^*]$. Also, $\underline{\nu}(a^*) = a$, $\underline{\nu}(b^*) = b$, and $\underline{\nu}(I^*) \subseteq [a, b]$.

PT2 $\underline{\nu}$ is $r \geq 1$ times continuously differentiable in I^*.

PT3 For all $t^* \in I^*$, $\underline{\nu}'(t^*) = \frac{dt}{dt^*} \neq 0$.

2.4 Arc of a Curve and Implications of Allowability Conditions

We now define an arc of a curve.

Definition 2.4.1 An *arc of a curve* is a point set M in $\mathbb{R}^3$ that is representable by an allowable representation $\vec{\underline{f}}(t)$. If multiple allowable representations exist, then those that are transformable from one to another via allowable parametric transformations are considered equivalent and represent the same arc. Those that are not, are considered to represent different arcs, even though the point set M is the same.

We now consider the implications of the conditions for allowable representations $\vec{\underline{f}}(t)$, and how they affect the definition of the arc of a curve.

In particular, conditions C1 and C2 were chosen from our previous developmentary ideas about curves, and from the desire to apply calculus in our study. Now consider condition C3. If, for a particular t, there exists a $k \in \{1,2,3\}$, so that $\underline{f}'_k(t) \neq 0$, then either $\underline{f}'_k(t) > 0$ or $\underline{f}'_k(t) < 0$. From calculus, i.e. Ellis and Gulick [2, Theorem 7.4], we know that $\underline{f}_k$ is one-to-one locally near t. Hence, M will not consist solely of one point.

If we similarly consider the allowability conditions on parametric transformations, we can see that they were chosen so that if $\vec{\underline{f}}(t)$ is allowable, then $\vec{\underline{f}}(\vec{\underline{\nu}}(t^*)) = \vec{\underline{h}}(t^*)$ is allowable. Also, as a consequence of the condition PT3, the mapping $\underline{\nu}$ is one-to-one, so the inverse transformation $t^* = \underline{\nu}^{-1}(t)$ exists and is allowable.

We will now address the question of the relationship between the allowability conditions and the ideas developed about Jordan arcs discussed in Chapter 1.

Theorem 2.4.1 *If* $\vec{x} = \vec{\underline{\mathrm{f}}}(t)$ *is an allowable representation for the arc of a curve, then* $\vec{\underline{\mathrm{f}}}$ *is continuous.*

Proof: Let $\varepsilon > 0$. By condition C2, for all $i \in \{1, 2, 3\}$, $\underline{\mathrm{f}}_i$ is differentiable in I, so $\underline{\mathrm{f}}_i$ is continuous in I. Therefore, there exists a $\delta > 0$, such that if $|t_2 - t_1| < \delta$, then $|\underline{\mathrm{f}}_i(t_2) - \underline{\mathrm{f}}_i(t_1)| < \frac{\varepsilon}{2}$. Now

$$|\vec{\underline{\mathrm{f}}}(t_2) - \vec{\underline{\mathrm{f}}}(t_1)| = \sqrt{\sum_{i=1}^{3}(\underline{\mathrm{f}}_i(t_2) - \underline{\mathrm{f}}_i(t_1))^2} < \sqrt{\sum_{i=1}^{3}\frac{\varepsilon^2}{4}} = \sqrt{\frac{3\varepsilon^2}{4}} = \varepsilon\sqrt{\frac{3}{4}} < \varepsilon.$$

Hence, $\vec{\underline{\mathrm{f}}}$ is continuous. ∎

To consider bijectivity, we must consider arcs with no "multiple points".

Definition 2.4.2 A *multiple point of an arc* is a point of the point set M corresponding to several values of t.

Definition 2.4.3 A *simple arc* is an arc having no multiple points. In this case, if $\vec{x} = \vec{\underline{\mathrm{f}}}(t)$, then $\vec{\underline{\mathrm{f}}}$ is one-to-one.

Theorem 2.4.2 *A simple arc is a Jordan arc.*

Proof: Assume that we have a simple arc. Then there exists an allowable representation $\vec{x} = \vec{\underline{\mathrm{f}}}(t)$, where $\vec{\underline{\mathrm{f}}}$ is one-to-one and maps $I = [a, b]$ onto the point set M. Hence $\vec{\underline{\mathrm{f}}}$ is bijective. By Theorem 2.4.1, $\vec{\underline{\mathrm{f}}}$ is continuous. By Theorem 1.6.1, $\vec{\underline{\mathrm{f}}}$ is topological. By definition, $\vec{\underline{\mathrm{f}}}$ is a Jordan arc. ∎

Remark Not every Jordan arc is the arc of a curve. In particular, the mapping for a Jordan arc need not satisfy any differentiability requirements. Hence our study has somewhat more stringent requirements than just that of a Jordan arc.

The preceding discussion shows that the "allowability" criteria provide exactly the setting we desire, namely a dimension-preserving map that can be studied using calculus.

2.5 Some Curve Definitions

We now define a "curve" and present a list of some useful definitions and notation.

Definition 2.5.1 A *curve* is a point set M in $\mathbb{R}^3$ that is representable by one of a set of equivalent allowable representations of the form $\vec{\underline{f}}(t)$, where the domain interval I need not be closed or bounded, but if the values of the parameter are restricted to any closed and bounded subinterval of I, then one obtains an arc of a curve.

Definition 2.5.2 A curve is said to be a *closed curve* if there exists at least one allowable representation $\vec{\underline{f}}(t)$ such that $\vec{\underline{f}}$ is periodic, i.e. for all $t \in I$, there exists an $\omega > 0$ so that $\vec{\underline{f}}(t+\omega) = \vec{\underline{f}}(t)$.

Definition 2.5.3 A *plane curve* or *plane arc* is a curve or an arc whose points all lie in one plane.

Definition 2.5.4 A *curve of class* r, where *class* r can symbolized by $C^{(r)}$, is a curve that can be represented by an allowable representation $\vec{\underline{f}}(t) = \underline{f}_1(t)\widehat{\imath} + \underline{f}_2(t)\widehat{\jmath} + \underline{f}_3(t)\widehat{k}$, where at least one of $\underline{f}_1$, $\underline{f}_2$, or $\underline{f}_3$ is r times continuously differentiable on I, and the other components are $s \geq r$ times continuously differentiable on I.

Notation: For convenience, we will often symbolize the representation $\vec{\underline{f}}(t) = \underline{f}_1(t)\widehat{\imath} + \underline{f}_2(t)\widehat{\jmath} + \underline{f}_3(t)\widehat{k}$ by $\langle \underline{f}_1(t), \underline{f}_2(t), \underline{f}_3(t)\rangle$.

2.6 Curve Representations

Before launching into the actual study of curves, it is worth mentioning some other common representations of curves. By condition C3, for every t, there exists a $k \in \{1,2,3\}$, so that $\underline{f}_k'(t) \neq 0$, and hence $\underline{f}_k$ is locally one-to-one near t. Thus near t, $\underline{f}_k$ is locally invertible, so that $t = \underline{f}_k^{-1}(x_k)$. Thus for all $t \in \tilde{I} \subseteq I$, the parameter can be eliminated. For example, suppose that for some $\tilde{t}$, $\underline{f}_1$ is locally invertible, then $t = \underline{f}_1^{-1}(x_1)$. Hence

$$x_2 = \underline{f}_2(t) = \underline{f}_2(\underline{f}_1^{-1}(x_1)) = \underline{f}_{x_2}(x_1).$$

Similarly, $x_3 = \underline{f}_{x_3}(x_1)$. Thus other common representations of curves are

$$\begin{cases} x_2 = \underline{f}_{x_2}(x_1) \\ x_3 = \underline{f}_{x_3}(x_1) \end{cases}$$
$$\begin{cases} x_1 = \underline{f}_{x_1}(x_2) \\ x_3 = \underline{f}_{x_3}(x_2) \end{cases}$$
$$\begin{cases} x_1 = \underline{f}_{x_1}(x_3) \\ x_2 = \underline{f}_{x_2}(x_3) \end{cases} .$$

Now under certain allowability conditions the equation $\underline{F}(x_1,x_2,x_3) = 0$ represents a surface in $\mathbb{R}^3$, so the point set

$$\begin{cases} \underline{F}(x_1,x_2,x_3) = 0 \\ \underline{G}(x_1,x_2,x_3) = 0 \end{cases}$$

could also be used to represent a curve, being the intersection of two surfaces.

For our purposes, we will use the representation $\vec{\underline{f}}(t)$, because all three coordinates play the same role. Furthermore, it is also more general, because a curve represented by $\vec{\underline{f}}(t)$ can not always be represented completely by the other representations. See Kreyszig [5] for details.

Chapter 3

$C^{(r)}$ curves, where $r \geq 1$

3.1 Introduction

In the chapters that follow, we summarize all of the major facts and results about curves. Each subsequent chapter will, in general, add an additional restriction to the family of curves that we consider. Therefore, the results from any chapter are typically applicable to the curves studied in following chapters. Since most of the results in these chapters should be familiar, they are often stated, rather than developed.

3.2 Arc Length

Theorem 3.2.1 *For $t \in [a, b]$, let $\vec{x} = \vec{\underline{f}}(t)$ be an allowable representation of an arc C of a $C^{(r)}$ curve, where $r \geq 1$. In that case, the arc C has a defined length s given by*

$$s = \int_a^b \sqrt{\sum_{i=1}^{3} (\underline{f}_i'(t))^2}\, dt = \int_a^b \sqrt{\vec{\underline{f}}'(t) \cdot \vec{\underline{f}}'(t)}\, dt.$$

Also, s is independent of the choice of representation $\vec{\underline{f}}(t)$.

Proof: See Kreyszig [5, Theorem 9.1]. ▮

Definition 3.2.1 The *arc length function,* $\underline{s}$, is defined by letting $t_0 \in [a, b]$, and then letting

$$\underline{s}(t) = \int_{t_0}^{t} \sqrt{\vec{\underline{f}}'(t) \cdot \vec{\underline{f}}'(t)}\, dt.$$

If $t > t_0$, then $\underline{s}(t) > 0$ and is equal to the length of the arc from $\vec{\underline{f}}(t_0)$ to $\vec{\underline{f}}(t)$ of C; if $t < t_0$, then $\underline{s}(t) < 0$ and the length of the arc is $-\underline{s}(t)$.

We note that by the Fundamental Theorem of Calculus,

$$\underline{s}'(t) = \sqrt{\vec{\underline{f}}'(t) \cdot \vec{\underline{f}}'(t)} = |\vec{\underline{f}}'(t)|, \tag{3.1}$$

so

$$(\underline{s}'(t))^2 = \vec{\underline{f}}'(t) \cdot \vec{\underline{f}}'(t) = \sum_{i=1}^{3} (\underline{f}_i'(t))^2. \tag{3.2}$$

Definition 3.2.2 The *element of arc* or *linear element* of C is ds, where if $s = \underline{s}(t)$ and $\vec{x} = \vec{\underline{f}}(t)$, then by equation 3.2,

$$ds^2 = d\vec{x} \cdot d\vec{x} = \sum_{i=1}^{3} d{x_i}^2,$$

where $ds = \underline{s}'(t)\,dt$, $d\vec{x} = \vec{\underline{f}}'(t)\,dt$, and $dx_i = \underline{f}_i'(t)\,dt$.

By Kreyszig [5], the equation $s = \underline{s}(t)$ is an allowable change of parameter. In particular, a $C^{(r)}$ curve, with $r \geq 1$, permits an allowable representation $\vec{\underline{f}}(s)$, where the point $s = 0$ may be chosen arbitrarily. s is the *arc length parameter* and is sometimes called the *natural parameter*. Use of the letter s will imply arc length parameter. A curve with an arc length parameterization is also called a unit speed curve.

3.3 Unit Tangent Vector

Definition 3.3.1 Let C be a $C^{(r)}$ curve, with $r \geq 1$. Then, the *unit tangent vector*, $\vec{t}$, to C at the point $\vec{x} = \vec{\underline{f}}(s)$ is defined by letting

$$\vec{t} = \frac{d\vec{x}}{ds} = \vec{\underline{f}}'(s) = \lim_{h \to 0} \frac{\vec{\underline{f}}(s+h) - \vec{\underline{f}}(s)}{h}.$$

Properties of the Unit Tangent Vector

T1 $\vec{t}$ is tangent to the curve C at $\vec{\underline{f}}(s)$.

T2 For all s, we know that $\vec{t}$ exists, because $\vec{\underline{f}}'(s)$ exists, since we have a $C^{(r)}$ curve, with $r \geq 1$.

T3 For all s, we know that $\vec{t} \neq \vec{0}$, since $\vec{\underline{f}}'(s) \neq \vec{0}$ by allowability condition C3 of Chapter 2.

T4 $|\vec{t}| = 1$. This is true since

$$\vec{t} \cdot \vec{t} = \vec{\underline{f}}'(s) \cdot \vec{\underline{f}}'(s) = \left(\vec{\underline{f}}'(t)\frac{dt}{ds}\right) \cdot \left(\vec{\underline{f}}'(t)\frac{dt}{ds}\right) = \frac{\vec{\underline{f}}'(t) \cdot \vec{\underline{f}}'(t)}{(\underline{s}'(t))^2}$$

which is equal to 1 by Equation 3.2.

T5 For a general parameter t,

$$\vec{t} = \frac{\vec{\underline{f}}'(t)}{|\vec{\underline{f}}'(t)|} = \frac{d\vec{x}}{ds}.$$

Theorem 3.3.1 *Let C be a $C^{(r)}$ curve, with $r \geq 1$. If $\vec{t}$ is constant, the curve is a straight line.*

Proof: Let $\vec{t}$ be a constant unit vector $\vec{a}$, so $\vec{t} = \vec{a}$. Therefore $\vec{\underline{f}}'(s) = \vec{a}$, so upon integrating, we have $\vec{\underline{f}}(s) = \vec{a}\,s + \vec{b}$, where $\vec{b}$ is constant vector of integration. This is the representation for a straight line. ∎

3.4 Tangent Line and Normal Plane

Definition 3.4.1 Let C be a $C^{(r)}$ curve, with $r \geq 1$. Then the *tangent line* to C at the point $\vec{x} = \vec{\underline{f}}(s)$ is the line passing through the associated unit tangent vector at the same point.

Definition 3.4.2 Let C be a $C^{(r)}$ curve, with $r \geq 1$. Then the *normal plane* to C at the point $\vec{x} = \vec{\underline{f}}(s)$ is the plane perpendicular to the associated unit tangent vector at the same point.

From the above definitions of the tangent line and normal plane, we can readily obtain their equations. In particular, the equation for the tangent line having position vector $\vec{y}_t$ is given by

$$\vec{y}_t = \vec{x} + u\,\vec{t}, \tag{3.3}$$

where u is a parameter. An equation for the normal plane having position vector $\vec{z}_t$ is given by

$$(\vec{z}_t - \vec{x}) \cdot \vec{t} = 0, \tag{3.4}$$

since $\vec{z}_t - \vec{x}$ lies in the normal plane and is perpendicular to $\vec{t}$.

Chapter 4

$C^{(r)}$ curves, where $r \geq 2$

4.1 Curvature Vector

Definition 4.1.1 Let C be a $C^{(r)}$ curve, with $r \geq 2$. The *curvature vector,* $\vec{k}$, to C at the point $\vec{x} = \vec{\underline{f}}(s)$ is defined by letting

$$\vec{k} = \frac{d\vec{t}}{ds} = \frac{d}{ds}\left(\vec{\underline{f}}'(s)\right) = \vec{\underline{f}}''(s).$$

Remark Since we have a $C^{(r)}$ curve, with $r \geq 2$, $\vec{\underline{f}}''(s)$ exists. So, in particular, $\vec{k}$ will always exist.

4.2 Osculating Plane

Definition 4.2.1 Let C be a $C^{(r)}$ curve, with $r \geq 2$. The *osculating plane* to C at the point $\vec{x} = \vec{\underline{f}}(s)$ is the plane spanned by $\vec{t}$ and $\vec{k}$ at the same point, provided that $\vec{t}$ and $\vec{k}$ are linearly independent.

One may ask the question as to what is known if $\vec{t}$ and $\vec{k}$ are linearly dependent. This is answered by the following theorem.

Theorem 4.2.1 *Let C be a $C^{(r)}$ curve, with $r \geq 2$. If the unit tangent vector, $\vec{t}$, and the curvature vector, $\vec{k}$, for a curve C are linearly dependent at every point of C, then C is a straight line.*

Proof: Kreyszig [5, Problem 11.1] ∎

Using the notation $[\vec{u}, \vec{v}, \vec{w}]$ for the triple scalar product of vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, i.e. $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})(\vec{u} \times \vec{v}) \cdot \vec{w}$, we can write the equation for the osculating plane.

An equation for the osculating plane having position vector $\vec{z}_b$ is given by

$$[\vec{z}_b - \vec{x}, \vec{t}, \vec{k}] = 0, \tag{4.1}$$

which can be rewritten in terms of the original function and a general parameter, to give us

$$[\vec{z}_b - \vec{\underline{f}}(t), \vec{\underline{f}}'(t), \vec{\underline{f}}''(t)] = 0. \tag{4.2}$$

Remark By Kreyszig [5], the osculating plane is the limiting position of a plane passing through three points P, P_1, P_2 of C, if P_1 and P_2 both tend to P.

4.3 Curvature

Definition 4.3.1 Let C be a $C^{(r)}$ curve, with $r \geq 2$. The *curvature*, κ, of C at the point $\vec{x} = \vec{\underline{f}}(s)$ is defined by letting

$$\kappa = |\vec{k}| = \left|\frac{d\vec{t}}{ds}\right| = \left|\vec{\underline{f}}''(s)\right|.$$

Remark Since we have a $C^{(r)}$ curve, with $r \geq 2$, we see that κ exists at every point $\vec{x} = \vec{\underline{f}}(s)$. Furthermore, by definition, $\kappa \geq 0$ and will be a continuous function of s.

Theorem 4.3.1 *Let C be a $C^{(r)}$ curve, with $r \geq 2$. Let the curvature, κ, of C be zero everywhere. Then C is a straight line.*

Proof: Let $\kappa = 0$. Then $\left|\frac{d\vec{t}}{ds}\right| = 0$. Therefore $\frac{d\vec{t}}{ds} = \vec{0}$, and so $\vec{t} = \vec{a}$, where $\vec{a}$ is a constant vector. By Theorem 3.3.1, the curve is a straight line. ∎

Chapter 5

$C^{(r)}$ curves, where $r \geq 2$ and $\kappa \neq 0$

5.1 Unit Principal Normal Vector

Definition 5.1.1 Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0.$ The *unit principal normal vector* , $\vec{p}$, to C at the point $\vec{x} = \vec{\underline{f}}(s)$ is defined by letting

$$\vec{p} = \frac{\vec{k}}{\left|\vec{k}\right|} = \frac{\frac{d\vec{t}}{ds}}{\left|\frac{d\vec{t}}{ds}\right|} = \frac{\vec{\underline{f}}''(s)}{\left|\vec{\underline{f}}''(s)\right|}.$$

Properties of the Unit Principal Normal Vector

P1 $\vec{p}$ exists and is nonzero, because $\vec{\underline{f}}''(s)$ exists and is nonzero, since we have a $C^{(r)}$ curve, where $r \geq 2$ and $\kappa \neq 0$.

P2 $\vec{p}$ is perpendicular to $\vec{t}$.

P3 $|\vec{p}| = 1$, by definition.

P4 $\vec{p}$ is independent of the orientation of the curve.

Remark Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0$. By Definition 5.1.1, at each point $\vec{x} = \vec{\underline{f}}(s)$ to C, we have that

$$\frac{d\vec{t}}{ds} = \kappa\vec{p}. \tag{5.1}$$

5.2 Formulae for Curvature

By Kreyszig [5], we have the following formulae for curvature:

$$\kappa = |\vec{\underline{f}}'(s) \times \vec{\underline{f}}''(s)|. \tag{5.2}$$

$$\kappa = \frac{|\vec{\underline{f}}'(t) \times \vec{\underline{f}}''(t)|}{\left|\vec{\underline{f}}'(t)\right|^3}. \tag{5.3}$$

Remark These formulae were derived under the assumption that $\vec{p}$ exists, so they are placed here in this chapter, rather than Chapter 4.

5.3 Principal Normal Line and Rectifying Plane

Definition 5.3.1 Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0$. The *principal normal line* to C at the point $\vec{x} = \vec{\underline{f}}(s)$ is the line passing through the associated unit principal normal vector at the same point.

Definition 5.3.2 Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0$. The *rectifying plane* to C at the point $\vec{x} = \vec{\underline{f}}(s)$ is the plane perpendicular to the associated unit principal normal vector at the same point.

From the above definitions of the principal normal line and rectifying plane, we can obtain their equations. In particular, the equation for the principal normal line having position vector $\vec{y}_p$ is given by

$$\vec{y}_p = \vec{x} + u\vec{p}, \tag{5.4}$$

where u is a parameter. The equation for the rectifying plane having position vector $\vec{z}_p$ is given by

$$(\vec{z}_p - \vec{x}) \cdot \vec{p} = 0, \tag{5.5}$$

since $\vec{z}_p - \vec{x}$ lies in the rectifying plane and is perpendicular to $\vec{p}$.

5.4 Radius of Curvature and Center of Curvature

Definition 5.4.1 Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0$. The *radius of curvature*, ρ, of C at the point $\vec{x} = \vec{\underline{f}}(s)$ is defined by letting

$$\rho = \frac{1}{\kappa}.$$

Definition 5.4.2 Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0$. The *center of curvature* of C for a point $\vec{x} = \vec{\underline{\mathrm{f}}}(s)$ is the point on the positive ray of $\vec{p}$ at a distance ρ from the point $\vec{x}$.

By definition, we see that the equation for the position vector $\vec{y}_\kappa$ to the center of curvature is given by

$$\vec{y}_\kappa = \vec{x} + \rho\vec{p} = \vec{x} + \rho^2\vec{k} = \vec{\underline{\mathrm{f}}}(s) + \rho^2\vec{\underline{\mathrm{f}}}''(s). \tag{5.6}$$

5.5 Unit Binormal Vector, Moving Trihedron, Binormal Line

Definition 5.5.1 Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0$. The *unit binormal vector*, $\vec{b}$, to C at the point $\vec{x} = \vec{\underline{\mathrm{f}}}(s)$ is defined by letting

$$\vec{b} = \vec{t} \times \vec{p}.$$

Properties of the Unit Binormal Vector

B1 $\vec{b}$ exists and is nonzero, because both $\vec{t}$ and $\vec{p}$ exist and are nonzero.

B2 $\vec{b}$ is perpendicular to both $\vec{t}$ and $\vec{p}$, by definition.

B3 $|\vec{b}| = 1$, because

$$|\vec{b}| = |\vec{t} \times \vec{p}| = |\vec{t}|\,|\vec{p}| \sin\frac{\pi}{2} = 1.$$

B4 For a general parameter t,

$$\vec{b} = \frac{\vec{\underline{\mathrm{f}}}'(t) \times \vec{\underline{\mathrm{f}}}''(t)}{|\vec{\underline{\mathrm{f}}}'(t) \times \vec{\underline{\mathrm{f}}}''(t)|}$$

B5 $\{\vec{t}, \vec{p}, \vec{b}\}$ form a right-handed set of vectors.

Definition 5.5.2 Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0$. The *moving trihedron* to C at the point $\vec{x} = \vec{\underline{\mathrm{f}}}(s)$ is the triple $\{\vec{t}, \vec{p}, \vec{b}\}$.

Remark Since the moving trihedron is a set of three mutually orthogonal unit vectors, the moving trihedron for a point $\vec{x} = \vec{\underline{\mathrm{f}}}(s)$ may serve as a basis for $\mathbb{R}^3$.

Definition 5.5.3 Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0$. The *binormal line* to C at the point $\vec{x} = \vec{\underline{\mathrm{f}}}(s)$ is the line passing through the associated unit binormal vector at the same point.

The equation for the binormal line having position vector $\vec{y}_b$ is given by

$$\vec{y}_b = \vec{x} + u\vec{b}, \tag{5.7}$$

where u is a parameter.

5.6 Some Results About Plane Curves and Osculating Planes

Theorem 5.6.1 *Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0$. The osculating plane to C at the point $\vec{x} = \vec{\underline{f}}(s)$ is the plane spanned by $\vec{t}$ and $\vec{p}$. The normal plane to C at the point $\vec{x} = \vec{\underline{f}}(s)$ is the plane spanned by $\vec{p}$ and $\vec{b}$. The rectifying plane to C at the point $\vec{x} = \vec{\underline{f}}(s)$ is the plane spanned by $\vec{t}$ and $\vec{b}$.*

Proof: By Definition 4.2.1, the osculating plane is spanned by $\vec{t}$ and $\vec{k}$. Since $\vec{k} = \kappa \vec{p}$, we have that the osculating plane is spanned by $\vec{t}$ and $\vec{p}$. Now the unit vectors in the moving trihedron are mutually orthogonal, so by Definitions 3.4.2 and 5.3.2, we have the normal plane spanned by $\vec{p}$ and $\vec{b}$, and the rectifying plane spanned by $\vec{t}$ and $\vec{b}$. ∎

By Theorem 5.6.1, we can come up with an alternate method for finding the equation of the osculating plane that parallels that of Equations 3.4 and 5.5 for the normal plane and the rectifying plane. In particular, the osculating plane having position vector $\vec{z}_b$ is given by

$$(\vec{z}_b - \vec{x}) \cdot \vec{b} = 0, \tag{5.8}$$

since $\vec{z}_b - \vec{x}$ lies in the osculating plane and is perpendicular to $\vec{b}$.

We now consider a few results about plane curves.

Theorem 5.6.2 *Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0$. $\vec{b}$ is constant on a curve C if and only if C is a plane curve.*

Proof: Suppose that $\vec{b}$ is constant on a curve C. By P2, $\vec{b} \cdot \vec{t} = 0$. Thus, $\vec{b} \cdot \vec{\underline{f}}'(s) = 0$. Now integrating, where $\vec{b}$ is constant, we have $\vec{b} \cdot \vec{\underline{f}}(s) = a_1$, where a_1 is a constant of integration. Thus, the curve C lies in a plane perpendicular to constant unit vector $\vec{b}$, and so is a plane curve. Conversely, suppose that C is a plane curve. Then $\vec{c} \cdot \vec{\underline{f}}(s) = a_2$, for some constant unit vector $\vec{c}$ that is perpendicular to the plane in which C lies, and a_2 is a constant. Differentiating, with $\vec{c}$ constant, we get $\vec{c} \cdot \vec{\underline{f}}'(s) = 0$. Hence $\vec{c} \cdot \vec{t} = 0$. Differentiating again, we get $\vec{c} \cdot \frac{d\vec{t}}{ds} = 0$. By Equation 5.1, we have $\vec{c} \cdot \kappa \vec{p} = 0$. Since $\kappa \neq 0$, we have $\vec{c} \cdot \vec{p} = 0$. Thus we have $\vec{c}$ perpendicular to both $\vec{t}$ and $\vec{p}$, and so $\vec{c}$ must lie in the direction of $\vec{b}$. Hence, there is a scalar α, so that $\vec{c} = \alpha \vec{b}$. Now $\vec{c}$ and $\vec{b}$ are unit vectors, so $\alpha = \pm 1$ and $\vec{b} = \pm \vec{c}$ is a constant vector. Thus $\vec{b} = \frac{1}{\alpha} \vec{c}$, and we have that $\vec{b}$ is constant. ∎

Remark We notice here that if C is a plane curve, then $\vec{b}$ is perpendicular to the plane of the curve.

Theorem 5.6.3 *Let C be a $C^{(r)}$ curve, with $r \geq 2$ and $\kappa \neq 0$. The osculating plane of a plane curve coincides with the plane of the curve.*

Proof: Suppose that C is a plane curve. By Theorem 5.6.2, $\vec{b}$ is constant and $\vec{b}$ is perpendicular to the plane of the curve. Hence, for every point $\vec{x} = \vec{\underline{f}}(s)$,

$$\vec{b} \cdot \vec{\underline{f}}(s) = a,$$

where a is a constant. Now by Equation 5.8, the position vector $\vec{z}_b$ to the osculating plane satisfies the equation $(\vec{z}_b - \vec{x}) \cdot \vec{b} = 0$. Hence,

$$\vec{z}_b \cdot \vec{b} - \vec{\underline{f}}(s) \cdot \vec{b} = 0.$$

Therefore,

$$\vec{z}_b \cdot \vec{b} = \vec{b} \cdot \vec{\underline{f}}(s) = a.$$

Thus, any point in the osculating plane lies in a plane perpendicular to the constant vector $\vec{b}$, but this is the plane in which the curve C lies. ∎

5.7 Determining the Equation of a Plane Curve from Curvature

Let C be a $C^{(r)}$ plane curve, with $r \geq 2$ and $\kappa \neq 0$. Let us assume that the plane curve lies in the xy plane. Let the plane curve have the allowable representation $\vec{y} = \vec{\underline{f}}_y(s)$. In this section, we show how the equation of a plane curve can be determined up to location and orientation by knowing the formula for curvature.

By Theorem 5.6.3, the osculating plane is the xy plane. Since $\vec{t}_y$ lies in the osculating plane, it makes an angle θ with the x-axis. Hence,

$$\vec{t}_y = \langle \cos\,\theta, \sin\,\theta, 0 \rangle\,. \tag{5.9}$$

Differentiating Equation 5.9, we get

$$\frac{d\vec{t}_y}{d\theta} = \langle -\sin\,\theta, \cos\,\theta, 0 \rangle. \tag{5.10}$$

By inspection, we see that $\vec{t}_y \cdot \frac{d\vec{t}_y}{d\theta} = 0$, so that $\frac{d\vec{t}_y}{d\theta}$ must be $\pm\vec{p}_y$. Therefore, by Equation 5.10,

$$\frac{d\vec{t}_y}{ds} = \left(\frac{d\vec{t}_y}{d\theta}\right)\left(\frac{d\theta}{ds}\right) = \left(\pm\frac{d\theta}{ds}\right)\vec{p}_y. \tag{5.11}$$

By Equation 5.1, we have

$$\kappa_y \vec{p}_y = \left(\pm\frac{d\theta}{ds}\right)\vec{p}_y.$$

Hence,

$$\kappa_y = \left|\frac{d\theta}{ds}\right|. \tag{5.12}$$

Since $\kappa_y \neq 0$, and is a continuous function of s, we may choose an orientation of the plane curve, so that

$$\kappa_y = \frac{d\theta}{ds}. \tag{5.13}$$

By Equation 5.13, we have

$$\theta = \underline{\theta}(s) = \int_{s_0}^{s} \underline{\kappa}_y(\gamma)\, d\gamma. \tag{5.14}$$

Now,

$$\vec{t}_y = \frac{d\vec{y}}{ds};$$

so we have

$$\vec{y} = \vec{\underline{\mathrm{f}}}_y(s) = \int_{s_0}^{s} \vec{\underline{\mathrm{t}}}_y(\gamma)\, d\gamma. \tag{5.15}$$

Substituting Equation 5.9 into Equation 5.15, we get

$$\vec{\underline{\mathrm{f}}}_y(s) = \left\langle \int_{s_0}^{s} \cos\, \underline{\theta}(\gamma)\, d\gamma, \int_{s_0}^{s} \sin\, \underline{\theta}(\gamma)\, d\gamma, 0 \right\rangle \tag{5.16}$$

Substituting Equation 5.14 into Equation 5.16, we get

$$\vec{\underline{\mathrm{f}}}_y(s) = \left\langle \int_{s_0}^{s} \left[\cos\left(\int_{\gamma_0}^{\gamma} \underline{\kappa}_y(\eta)\, d\eta\right)\right] d\gamma, \int_{s_0}^{s} \left[\sin\left(\int_{\gamma_0}^{\gamma} \underline{\kappa}_y(\eta)\, d\eta\right)\right] d\gamma, 0 \right\rangle. \tag{5.17}$$

Remark Equation 5.17 gives the representation for a plane curve in terms of curvature. However, we note that orientation and location needed to be specified. Note that in Equation 5.14, we have specified that $\underline{\theta}(s_0) = 0$. This sets the orientation of the curve, because by Equations 5.9 and 5.10, we have that $\vec{\underline{\mathrm{t}}}_y(s_0) = \langle 1, 0, 0\rangle$ and $\vec{\underline{\mathrm{p}}}_y(s_0) = \langle 0, 1, 0\rangle$. Furthermore, by Equation 5.15, we have specified that $\vec{\underline{\mathrm{f}}}_y(s_0) = 0$. This sets the location of the curve, namely, that the curve passes through the origin at the initial parameter value.

Chapter 6

$C^{(r)}$ curves, where $r \geq 3$ and $\kappa \neq 0$

6.1 Derivative of $\vec{b}$ and Torsion

Theorem 6.1.1 *Let C be a $C^{(r)}$ curve, with $r \geq 3$ and $\kappa \neq 0$. For all points $\vec{x} = \vec{\underline{f}}(s)$ on C, $\frac{d\vec{b}}{ds}$ exists.*

Proof: By Property B1 from Section 5.5, for all points $\vec{x} = \vec{\underline{f}}(s)$, $\vec{b}$ exists and is nonzero. By definition, $\vec{b} = \vec{t} \times \vec{p}$, so $\vec{b} = \vec{t} \times \frac{\vec{k}}{\kappa}$. Thus we have

$$\vec{b} = \frac{\vec{\underline{f}}'(s) \times \vec{\underline{f}}''(s)}{\left|\vec{\underline{f}}''(s)\right|}. \tag{6.1}$$

Since we have a $C^{(r)}$ curve, with $r \geq 3$ and $\kappa \neq 0$, by Equation 6.1, we see that $\frac{d\vec{b}}{ds}$ exists. ∎

Definition 6.1.1 Let C be a $C^{(r)}$ curve, with $r \geq 3$ and $\kappa \neq 0$. Then the *torsion*, τ, of C at the point $\vec{x} = \vec{\underline{f}}(s)$ is defined by letting

$$\tau = -\vec{p} \cdot \frac{d\vec{b}}{ds}.$$

Remark Since both $\vec{p}$ and $\frac{d\vec{b}}{ds}$ exist for a $C^{(r)}$ curve, with $r \geq 2$, we can see that τ exists at every point $\vec{x} = \vec{\underline{f}}(s)$.

Theorem 6.1.2 *Let C be a $C^{(r)}$ curve, with $r \geq 3$ and $\kappa \neq 0$. At each point $\vec{x} = \vec{\underline{f}}(s)$ to C, we have that*

$$\frac{d\vec{b}}{ds} = -\tau\vec{p}. \tag{6.2}$$

Proof: Since $\vec{b} \cdot \vec{b} = 1$, differentiating we get

$$\frac{d\vec{b}}{ds} \cdot \vec{b} + \vec{b} \cdot \frac{d\vec{b}}{ds} = 0.$$

Thus

$$2\vec{b} \cdot \frac{d\vec{b}}{ds} = 0,$$

or

$$\vec{b} \cdot \frac{d\vec{b}}{ds} = 0.$$

Likewise, since $\vec{b} \cdot \vec{t} = 0$, differentiating we get

$$\frac{d\vec{b}}{ds} \cdot \vec{t} + \vec{b} \cdot \frac{d\vec{t}}{ds} = 0.$$

By Equation 5.1, we have

$$\frac{d\vec{b}}{ds} \cdot \vec{t} + \vec{b} \cdot \kappa\vec{p} = 0.$$

Hence,

$$\vec{t} \cdot \frac{d\vec{b}}{ds} = -\kappa(\vec{b} \cdot \vec{p}) = 0.$$

Since the moving trihedron may be used as a basis for $\mathbb{R}^3$, we may write

$$\frac{d\vec{b}}{ds} = a\vec{t} + b\vec{p} + c\vec{b},$$

for some constants a, b, and c. Now $a = \vec{t} \cdot \frac{d\vec{b}}{ds} = 0$, and $c = \vec{b} \cdot \frac{d\vec{b}}{ds} = 0$ as derived, and $b = \vec{p} \cdot \frac{d\vec{b}}{ds} = -\tau$, by Definition 6.1.1. Hence we have

$$\frac{d\vec{b}}{ds} = -\tau\vec{p}.$$

∎

6.2 Properties of Torsion

Let C be a $C^{(r)}$ curve, with $r \geq 3$ and $\kappa \neq 0$. The following results can be shown to be true.

1. If $\tau = 0$ at every point $\vec{x} = \vec{\underline{f}}(s)$ of C, then C is a plane curve.
2. If $\tau > 0$ at every point $\vec{x} = \vec{\underline{f}}(s)$ of C, then C is a right-handed curve.
3. If $\tau < 0$ at every point $\vec{x} = \vec{\underline{f}}(s)$ of C, then C is a left-handed curve.

6.3 Formulae for Torsion

By Kreyszig [5], we have the following formulae for torsion:

$$\tau = \frac{[\vec{\underline{\mathrm{f}}}'(s), \vec{\underline{\mathrm{f}}}''(s), \vec{\underline{\mathrm{f}}}'''(s)]}{\vec{\underline{\mathrm{f}}}''(s) \cdot \vec{\underline{\mathrm{f}}}''(s)}. \tag{6.3}$$

$$\tau = \frac{[\vec{\underline{\mathrm{f}}}'(t), \vec{\underline{\mathrm{f}}}''(t), \vec{\underline{\mathrm{f}}}'''(t)]}{\left|\vec{\underline{\mathrm{f}}}'(t) \times \vec{\underline{\mathrm{f}}}''(t)\right|^2}. \tag{6.4}$$

6.4 Frenet-Serret Formulae

Theorem 6.4.1 *Let C be a $C^{(r)}$ curve, with $r \geq 3$ and $\kappa \neq 0$. At each point $\vec{x} = \vec{\underline{\mathrm{f}}}\,(s)$ to C, we have that*

$$\frac{d\vec{p}}{ds} = -\kappa \vec{t} + \tau \vec{b}. \tag{6.5}$$

Proof: Since $\vec{p} \cdot \vec{t} = 0$, differentiating we get

$$\frac{d\vec{p}}{ds} \cdot \vec{t} + \vec{p} \cdot \frac{d\vec{t}}{ds} = 0.$$

Thus

$$\vec{t} \cdot \frac{d\vec{p}}{ds} = -\vec{p} \cdot \frac{d\vec{t}}{ds} = -\vec{p} \cdot \kappa \vec{p} = -\kappa(\vec{p} \cdot \vec{p}) = -\kappa.$$

Since $\vec{p} \cdot \vec{p} = 1$, differentiating we get

$$\frac{d\vec{p}}{ds} \cdot \vec{p} + \vec{p} \cdot \frac{d\vec{p}}{ds} = 0.$$

Thus

$$2\vec{p} \cdot \frac{d\vec{p}}{ds} = 0,$$

or

$$\vec{p} \cdot \frac{d\vec{p}}{ds} = 0.$$

Since $\vec{p} \cdot \vec{b} = 0$, differentiating we get

$$\frac{d\vec{p}}{ds} x \cdot \vec{b} + \vec{p} \cdot \frac{d\vec{b}}{ds} = 0.$$

Thus
$$\vec{b} \cdot \frac{d\vec{p}}{ds} = -\vec{p} \cdot \frac{d\vec{b}}{ds} = \tau.$$
Since the moving trihedron may be used as a basis for $\mathbb{R}^3$, we may write
$$\frac{d\vec{p}}{ds} = d\,\vec{t} + e\vec{p} + f\,\vec{b}.$$
for some constants d, e, and f. Now $d = \vec{t} \cdot \frac{d\vec{p}}{ds} = -\kappa$, $e = \vec{p} \cdot \frac{d\vec{p}}{ds} = 0$, and $f = \vec{b} \cdot \frac{d\vec{p}}{ds} = \tau$ as derived. Hence we have
$$\frac{d\vec{p}}{ds} = -\kappa\,\vec{t} + \tau\,\vec{b}.$$

∎

Putting together Equation 5.1 and Theorems 6.5 and 6.2, we collectively have what is known as the *Frenet-Serret Formulae.*

Definition 6.4.1 Let C be a $C^{(r)}$ curve, with $r \geq 3$ and $\kappa \neq 0$. At each point $\vec{x} = \vec{\underline{f}}(s)$ to C, the *Frenet-Serret Formulae* hold. Namely,
$$\frac{d\vec{t}}{ds} = \kappa\vec{p}. \tag{6.6}$$
$$\frac{d\vec{p}}{ds} = -\kappa\,\vec{t} + \tau\,\vec{b}. \tag{6.7}$$
$$\frac{d\vec{b}}{ds} = -\tau\vec{p}. \tag{6.8}$$

We can write the Frenet-Serret Formulae in matrix form. By writing Equations 6.6-6.8, as
$$\frac{d}{ds}\begin{bmatrix} \vec{t} \\ \vec{p} \\ \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{p} \\ \vec{b} \end{bmatrix}. \tag{6.9}$$

Remark When the moving trihedron is used as a basis for $\mathbb{R}^3$, and vectors are written in this basis, if derivatives are taken, the Frenet-Serret Formulae allow for an immediate replacement of the derivatives of $\vec{t}$, $\vec{p}$, and $\vec{b}$ in terms of the vectors in the moving trihedron.

6.5 Fundamental Theorem of Curves

By Equation 6.9, we observe that we have a "matrix differential equation". It appears naively that if one could specify the curvature, κ, and torsion, τ, in this "matrix differential equation", and then solve it, a curve of interest could be uniquely identified. We did a similar thing for plane curves in Section 5.7. In fact, this is, in essence, true by the following theorem.

Theorem 6.5.1 (Fundamental Theorem of Curves) *If $\kappa = \underline{\kappa}(s)(> 0)$ and $\tau = \underline{\tau}(s)$ are continuous functions of s, defined on an interval $[0, a]$ for some constant a, then there exists a unique arc of a curve C having curvature κ and torsion τ, determined up to location and orientation in space.*

Proof: See Kreyszig [5, Theorem 20.1]. ∎

Remark Given that the arc of a curve can be reconstructed from κ and τ by Theorem 6.5.1, the set of equations

$$\begin{cases} \kappa = \underline{\kappa}(s)(> 0) \\ \tau = \underline{\tau}(s) \end{cases}$$

are called the *natural equations* for the arc of a curve.

Chapter 7

$C^{(r)}$ curves, where $r \geq 3$, $\kappa \neq 0$, and $\tau \neq 0$; and General Helices

7.1 General Helices and Lancret's Theorem

We now introduce the concept of a general helix.

Definition 7.1.1 Let C be a $C^{(r)}$ curve, with $r \geq 3$, $\kappa \neq 0$, and $\tau \neq 0$. C is called a *general helix* if its tangent line makes a constant angle with a fixed line in space.

An easy way to test to see if a curve is a general helix is by using Lancret's Theorem.

Theorem 7.1.1 (Lancret's Theorem) *Let C be a $C^{(r)}$ curve, with $r \geq 3$, $\kappa \neq 0$, and $\tau \neq 0$. Then C is a general helix if and only if $\frac{\tau}{\kappa}$ is constant for every point $\vec{x} = \vec{\underline{f}}(s)$ on C,*

Proof: Let C be a $C^{(r)}$ curve, with $r \geq 3$, $\kappa \neq 0$, and $\tau \neq 0$. Let C be a general helix. Then, by definition, the tangent line to C makes a constant angle with a fixed line in space. Let $\vec{c}$ be any unit vector pointing along the fixed line. Therefore $\vec{c}$ makes a constant angle β with $\vec{t}$, so that $\vec{c} \cdot \vec{t} = \cos\beta$. We now show that $\beta \neq 0, \pi$. Assume, by way of contradiction, that $\beta = 0$ or $\beta = \pi$. In that case, $\vec{c} \cdot \vec{t} = 1$ or $\vec{c} \cdot \vec{t} = -1$. This implies that $\vec{t} = \vec{c}$ or $\vec{t} = -\vec{c}$. In either case, $\kappa = \left|\frac{d\vec{t}}{ds}\right| = 0$. However, this is a contradiction. Hence, $\beta \neq 0, \pi$. We now also show that $\beta \neq \frac{\pi}{2}$. Assume by way of contradiction, that $\beta = \frac{\pi}{2}$. In this case, $\vec{c} \cdot \vec{t} = 0$. Integrating, with $\vec{c}$ constant, yields $\vec{c} \cdot \vec{\underline{f}}(s) = a$, where a is a constant. Hence C lies in a plane perpendicular to constant vector $\vec{c}$, and thus C is a plane curve. By Theorem 5.6.2, $\vec{b}$ is constant on C, so $\frac{d\vec{b}}{ds} = \vec{0}$. Thus $\tau = -\vec{p} \cdot \frac{d\vec{b}}{ds} = 0$, a contradiction. Hence, $\beta \neq \frac{\pi}{2}$. Thus $\beta \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. Differentiating $\vec{c} \cdot \vec{t} = \cos\beta$ with $\vec{c}$ constant yields $\vec{c} \cdot \frac{d\vec{t}}{ds} = 0$. By Equation 6.6, we have $\vec{c} \cdot \kappa\vec{p} = 0$; so with $\kappa \neq 0$, we have $\vec{c} \cdot \vec{p} = 0$. Thus $\vec{c}$ is perpendicular to $\vec{p}$, and thus lies in the rectifying plane. Since the rectifying plane is spanned by $\vec{t}$ and $\vec{b}$ by Theorem 5.6.1, and since $\beta \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$, $\vec{c}$ has nonzero components along both $\vec{t}$ and $\vec{b}$. Hence $\vec{c} \cdot \vec{b} > 0$ or $\vec{c} \cdot \vec{b} < 0$. Now define a vector $\vec{u}$ by letting

$$\vec{u} = \begin{cases} \vec{c}; & \text{if } \vec{c} \cdot \vec{b} > 0 \\ -\vec{c}; & \text{if } \vec{c} \cdot \vec{b} < 0 \end{cases} .$$

By definition, $\vec{u}$ lies along the given fixed line, lies in the rectifying plane, makes a constant angle $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ with $\vec{t}$, and has a positive component with $\vec{b}$ [Note: if $\vec{u} = \vec{c}$, then $\alpha = \beta$; if $\vec{u} = -\vec{c}$, then $\alpha = \pi - \beta$]. Hence $\vec{u} \cdot \vec{t} = \cos\alpha$, and so $\vec{u} = (\cos\alpha)\vec{t} + b\vec{b}$, for some $b > 0$. Now

$$|\vec{u}|^2 = |\pm\vec{c}|^2 = 1 = \cos^2\alpha + b^2.$$

Thus

$$b^2 = 1 - \cos^2\alpha = \sin^2\alpha.$$

Since $b > 0$, $b = \sin\alpha$. Hence

$$\vec{u} = \cos\alpha\,\vec{t} + \sin\alpha\,\vec{b}.$$

Now

$$\frac{d\vec{u}}{ds} = \frac{d}{ds}(\pm\vec{c}) = \vec{0}.$$

Thus

$$(\cos\alpha)\frac{d\vec{t}}{ds} + (\sin\alpha)\frac{d\vec{b}}{ds} = \vec{0}.$$

By Equations 6.6 and 6.8, we have

$$(\cos\alpha)\kappa\vec{p} + (\sin\alpha)(-\tau\vec{p}) = \vec{0}.$$

Hence $(\kappa\cos\alpha - \tau\sin\alpha)\vec{p} = \vec{0}$. Therefore $\kappa\cos\alpha = \tau\sin\alpha$. Since $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$, $\sin\alpha \neq 0$, so we have

$$\frac{\tau}{\kappa} = \frac{\cos\alpha}{\sin\alpha}.$$

Therefore $\frac{\tau}{\kappa}$ is constant. Conversely, let C be a $C^{(r)}$ curve, with $r \geq 3$, $\kappa \neq 0$, $\tau \neq 0$, and let $\frac{\tau}{\kappa}$ be constant. Therefore, there exists a constant c so that $\frac{\tau}{\kappa} = c$. Now define a vector $\vec{u}$ by letting

$$\vec{u} = \frac{c}{\sqrt{1+c^2}}\vec{t} + \frac{1}{\sqrt{1+c^2}}\vec{b}.$$

We notice that

$$|\vec{u}|^2 = \frac{c^2}{1+c^2} + \frac{1}{1+c^2} = \frac{1+c^2}{1+c^2} = 1,$$

so $\vec{u}$ is a unit vector. Now

$$\frac{d\vec{u}}{ds} = \frac{c}{\sqrt{1+c^2}}\frac{d\vec{t}}{ds} + \frac{1}{\sqrt{1+c^2}}\frac{d\vec{b}}{ds}.$$

By Equations 6.6 and 6.8, we have

$$\frac{d\vec{u}}{ds} = \frac{c}{\sqrt{1+c^2}}(\kappa\vec{p}) + \frac{1}{\sqrt{1+c^2}}(-\tau\vec{p}) = \frac{c\kappa - \tau}{\sqrt{1+c^2}}\vec{p} = \frac{(\frac{\tau}{\kappa})\kappa - \tau}{\sqrt{1+c^2}}\vec{p} = 0\vec{p} = \vec{0}.$$

Since $\frac{d\vec{u}}{ds} = \vec{0}$, $\vec{u}$ is constant and defines a fixed line in space. Now let θ be the angle between $\vec{u}$ and $\vec{t}$, so

$$\theta = \text{Cos}^{-1}\left(\frac{\vec{u}\cdot\vec{t}}{|\vec{u}||\vec{t}|}\right) = \text{Cos}^{-1}(\vec{u}\cdot\vec{t}) = \text{Cos}^{-1}\left(\frac{c}{\sqrt{1+c^2}}\right).$$

Since $c = \frac{\tau}{\kappa} \neq 0$ is constant, and $\frac{c}{\sqrt{1+c^2}} \in (-1,1)$, the tangent line makes a constant angle $\theta \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ with a fixed line in space. Therefore C is a general helix. ∎

7.2 A Class of General Helices

A special class of general helices that has properties that are easy to calculate is given in the following theorem. This theorem was postulated and proved by Heuvers [4].

Theorem 7.2.1 *Let C be a $C^{(r)}$ curve, with $r \geq 3$, $\kappa \neq 0$, $\tau \neq 0$, and let C have the representation $\vec{x} = \vec{\underline{f}}(t) = \langle \frac{bt^n}{n}, (\sqrt{2b})\frac{t^{n+1}}{n+1}, \frac{t^{n+2}}{n+2}\rangle$, where $b > 0$ is a constant, and $n \neq -2, -1, 0$. Then C is a general helix. Furthermore,*

$$\kappa = \tau = \frac{\sqrt{2b}}{t^{n-1}(t^2+b)^2}.$$

Proof: From the definition of $\vec{\underline{f}}(t)$, we have

$$\frac{d\vec{x}}{dt} = \vec{\underline{f}}'(t) = \langle bt^{n-1}, \sqrt{2b}t^n, t^{n+1}\rangle. \tag{7.1}$$

Letting $t_0 = 0$ arbitrarily, we can now find the arc length function $\underline{s}(t)$, by Definition 3.2.1. Namely,

$$\begin{aligned}
\underline{s}(t) &= \int_0^t \sqrt{\vec{\underline{f}}'(t)\cdot\vec{\underline{f}}'(t)}\,dt \\
&= \int_0^t \sqrt{b^2t^{2n-2} + 2bt^{2n} + t^{2n+2}}\,dt \\
&= \int_0^t t^{n-1}\sqrt{b^2 + 2bt^2 + t^4}\,dt \\
&= \int_0^t t^{n-1}(t^2+b)\,dt \\
&= \int_0^t (t^{n+1} + bt^{n-1})\,dt.
\end{aligned}$$

Hence,

$$\underline{s}(t) = \frac{t^{n+2}}{n+2} + \frac{bt^n}{n}. \tag{7.2}$$

Now, by Equation 7.1, we have

$$\frac{d^2\vec{x}}{dt^2} = \vec{\underline{f}}''(t) = \langle b(n-1)t^{n-2}, (\sqrt{2b})nt^{n-1}, (n+1)t^n\rangle. \tag{7.3}$$

Now let us use Equation 5.3 to calculate curvature, κ. We have

$$\begin{aligned}
\kappa &= \frac{|\vec{\underline{\mathrm{f}}}'(t) \times \vec{\underline{\mathrm{f}}}''(t)|}{\left|\vec{\underline{\mathrm{f}}}'(t)\right|^3} \\
&= \frac{|\left\langle bt^{n-1}, \sqrt{2b}t^n, t^{n+1}\right\rangle \times \left\langle b(n-1)t^{n-2}, (\sqrt{2b})nt^{n-1}, (n+1)t^n\right\rangle|}{\left|\left\langle bt^{n-1}, \sqrt{2b}t^n, t^{n+1}\right\rangle\right|^3} \\
&= \frac{|\left\langle t^{2n}\sqrt{2b}, -2bt^{2n-1}, bt^{2n-2}\sqrt{2b}\right\rangle|}{(t^{n+1}+bt^{n-1})^3} \\
&= \frac{t^{2n-2}\sqrt{2b}\sqrt{t^4+2bt^2+b^2}}{(t^{n+1}+bt^{n-1})^3} \\
&= \frac{t^{2n-2}\sqrt{2b}(t^2+b)}{(t^{n-1}(t^2+b))^3}.
\end{aligned}$$

Hence,

$$\kappa = \frac{\sqrt{2b}}{t^{n-1}(t^2+b)^2}. \tag{7.4}$$

Now, by Equation 7.3, we have

$$\frac{d^3\vec{x}}{dt^3} = \vec{\underline{\mathrm{f}}}'''(t) = \left\langle b(n-1)(n-2)t^{n-3}, (\sqrt{2b})n(n-1)t^{n-2}, n(n+1)t^{n-1}\right\rangle. \tag{7.5}$$

Now let us use Equation 6.4 to calculate torsion, τ. We have

$$\begin{aligned}
\tau &= \frac{[\vec{\underline{\mathrm{f}}}'(t), \vec{\underline{\mathrm{f}}}''(t), \vec{\underline{\mathrm{f}}}'''(t)]}{\left|\vec{\underline{\mathrm{f}}}'(t) \times \vec{\underline{\mathrm{f}}}''(t)\right|^2} \\
&= \frac{\left\langle t^{2n}\sqrt{2b}, -2bt^{2n-1}, bt^{2n-2}\sqrt{2b}\right\rangle \cdot \left\langle b(n-1)(n-2)t^{n-3}, (\sqrt{2b})n(n-1)t^{n-2}, n(n+1)t^{n-1}\right\rangle}{(t^{2n-2}\sqrt{2b}(t^2+b))^2} \\
&= \frac{2bt^{3n-3}\sqrt{2b}}{2b(t^2+b)^2t^{4n-4}}
\end{aligned}$$

Hence,

$$\tau = \frac{\sqrt{2b}}{t^{n-1}(t^2+b)^2}. \tag{7.6}$$

Dividing Equation 7.6 by Equation 7.4, we get $\frac{\tau}{\kappa} = 1$, and thus by Lancret's Theorem, C is a general helix. ∎

We now examine the correspondence between general helices and special associated plane curves, which can be considered to be generators of the general helix. The material in the subsequent sections was developed by Heuvers [4].

7.3 The Associated Generating Plane Curve For A General Helix

Let C be a $C^{(r)}$ curve, with $r \geq 3$, $\kappa \neq 0$, $\tau \neq 0$, and let C be a general helix.

By definition, and by Lancret's Theorem, there exists a constant angle $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ with $\vec{u} \cdot \vec{t} = \cos\alpha$ so that

$$\vec{u} = (\cos\alpha)\,\vec{t} + (\sin\alpha)\,\vec{b}. \tag{7.7}$$

Now define a curve $\vec{y} = \vec{\underline{f}}_y(s)$, where

$$\vec{\underline{f}}_y(s) = \frac{\vec{\underline{f}}(s) - \vec{\underline{f}}(s_0) - (s - s_0)(\cos\alpha)\vec{u}}{\sin\alpha}. \tag{7.8}$$

We now show that $\vec{\underline{f}}_y(s)$ is a plane curve lying in a plane perpendicular to $\vec{u}$. Differentiating Equation 7.8, we get

$$\vec{\underline{f}}'_y(s) = \frac{\vec{\underline{f}}'(s) - (\cos\alpha)\vec{u}}{\sin\alpha} = \frac{\vec{t} - (\cos\alpha)\vec{u}}{\sin\alpha}. \tag{7.9}$$

By Equation 7.9, we have

$$\vec{\underline{f}}'_y(s) \cdot \vec{u} = \frac{\vec{u} \cdot \vec{t} - (\cos\alpha)}{\sin\alpha} = \frac{\cos\alpha - \cos\alpha}{\sin\alpha} = 0. \tag{7.10}$$

Since $\vec{\underline{f}}'_y(s) \cdot \vec{u} = 0$, integrating each side with $\vec{u}$ constant yields

$$(\vec{\underline{f}}_y(s) - \vec{\underline{f}}_y(s_0)) \cdot \vec{u} = 0. \tag{7.11}$$

However, by Equation 7.8, $\vec{\underline{f}}_y(s_0) = \vec{0}$, so $\vec{\underline{f}}_y(s) \cdot \vec{u} = 0$. Therefore, $\vec{\underline{f}}_y(s)$ lies in a plane perpendicular to constant vector $\vec{u}$.

Remark Since $\vec{\underline{f}}_y(s_0) = \vec{0}$, the plane curve $\vec{\underline{f}}_y(s)$ passes through the origin.

Now let us calculate various quantities related to the associated plane curve.

By Equation 7.9, we have the tangent vector to $\vec{\underline{f}}_y(s)$. It is given by

$$\vec{t}_y = \frac{\vec{t} - (\cos\alpha)\vec{u}}{\sin\alpha}. \tag{7.12}$$

Rewriting Equation 7.12, we have

$$\vec{t} = (\sin\alpha)\,\vec{t}_y + (\cos\alpha)\vec{u}.$$

Differentiating, we get

$$\frac{d\vec{t}}{ds} = (\sin\alpha)\frac{d\vec{t}_y}{ds}.$$

By Equation 6.6, we have

$$\kappa\vec{p} = (\sin\alpha)\kappa_y\vec{p}_y.$$

Hence

$$\vec{p}_y = \vec{p} \text{ and } \kappa_y = \frac{\kappa}{\sin\alpha}. \tag{7.13}$$

Substituting Equation 7.7 into Equation 7.12, yields

$$\begin{aligned}
\vec{t}_y &= \frac{\vec{t} - (\cos\alpha)((\cos\alpha)\vec{t} + (\sin\alpha)\vec{b})}{\sin\alpha} \\
&= \frac{(1-\cos^2\alpha)\vec{t} - (\sin\alpha\cos\alpha)\vec{b}}{\sin\alpha} \\
&= \frac{(\sin^2\alpha)\vec{t} - (\sin\alpha\cos\alpha)\vec{b}}{\sin\alpha}.
\end{aligned}$$

Hence,

$$\vec{t}_y = (\sin\alpha)\vec{t} - (\cos\alpha)\vec{b}. \tag{7.14}$$

Now

$$\vec{b}_y = \vec{t}_y \times \vec{p}_y.$$

By Equations 7.14 and 7.13, we have

$$\vec{b}_y = ((\sin\alpha)\vec{t} - (\cos\alpha)\vec{b}) \times \vec{p} = (\sin\alpha)(\vec{t} \times \vec{p}) - (\cos\alpha)(\vec{b} \times \vec{p}) = (\sin\alpha)\vec{b} - (\cos\alpha)(-\vec{t}).$$

By Equation 7.7, we have

$$\vec{b}_y = \vec{u} = (\cos\alpha)\vec{t} + (\sin\alpha)\vec{b}. \tag{7.15}$$

Thus $\frac{d\vec{b}_y}{ds} = \vec{0}$, so $\tau_y = 0$. However, we already knew this, since $\vec{\underline{\mathrm{f}}}_y(s)$ is a plane curve.

Remark From Lancret's Theorem, the constant angle α satisfies the following relationships:

$$\frac{\tau}{\kappa} = \frac{\cos\alpha}{\sin\alpha} = c, \text{ where } c \text{ is constant.} \tag{7.16}$$

$$\cos\alpha = \frac{c}{\sqrt{1+c^2}}. \tag{7.17}$$

$$\sin\alpha = \frac{1}{\sqrt{1+c^2}}. \tag{7.18}$$

These relationships may be useful if α is not immediately obvious from context.

7.4 Using a Plane Curve to Generate a General Helix

Let $\vec{\underline{\mathrm{f}}}_y(s)$ be an allowable representation of a $C^{(r)}$ plane curve, with $r \geq 3$ and $\kappa_y \neq 0$. Also, let $\vec{\underline{\mathrm{f}}}_y(s_0) = 0$ so that the plane curve passes through the origin.

Since $\vec{\underline{\mathrm{f}}}_y(s)$ is a plane curve, $\tau_y = 0$, $\vec{b}_y$ is a constant vector $\vec{u}$, where $\vec{u}$ is perpendicular to the plane of the curve [By Theorem 5.6.3, the osculating plane is the plane of the curve, and $\vec{b}_y$ is perpendicular to it].

Let $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. Define

$$\vec{\underline{f}}(s) = \vec{\underline{f}}(s_0) + (\sin\alpha)\,\vec{\underline{f}}_y(s) + (s - s_0)(\cos\alpha)\vec{u}, \tag{7.19}$$

where $\vec{\underline{f}}(s_0)$ is an arbitrary constant vector.

We now show that $\vec{\underline{f}}(s)$ is a general helix. Differentiating Equation 7.19, yields

$$\vec{t} = \vec{\underline{f}}'(s) = (\sin\alpha)\,\vec{t}_y + (\cos\alpha)\vec{u}. \tag{7.20}$$

Differentiating Equation 7.20, we get

$$\frac{d\vec{t}}{ds} = (\sin\alpha)\frac{d\vec{t}_y}{ds}.$$

By Equation 6.6, we have

$$\kappa\vec{p} = (\sin\alpha)\kappa_y\vec{p}_y.$$

Hence

$$\vec{p} = \vec{p}_y \text{ and } \kappa = (\sin\alpha)\kappa_y. \tag{7.21}$$

Now $\vec{b} = \vec{t} \times \vec{p}$, so using Equations 7.20 and 7.21, we have

$$\begin{aligned}\vec{b} = \vec{t} \times \vec{p} &= ((\sin\alpha)\,\vec{t}_y + (\cos\alpha)\vec{u}) \times \vec{p}_y \\ &= (\sin\alpha)(\vec{t}_y \times \vec{p}_y) + (\cos\alpha)(\vec{b}_y \times \vec{p}_y) \\ &= (\sin\alpha)\,\vec{b}_y - (\cos\alpha)\,\vec{t}_y.\end{aligned}$$

Hence,

$$\vec{b} = -(\cos\alpha)\,\vec{t}_y + (\sin\alpha)\,\vec{b}_y = -(\cos\alpha)\,\vec{t}_y + (\sin\alpha)\vec{u}. \tag{7.22}$$

Differentiating Equation 7.22, we get

$$\frac{d\vec{b}}{ds} = (-\cos\alpha)\frac{d\vec{t}_y}{ds}.$$

By Equations 6.6 and 6.8, we have

$$-\tau\vec{p} = (-\cos\alpha)\kappa_y\vec{p}_y$$

By Equation 7.21, we have

$$\tau\vec{p} = \kappa_y(\cos\alpha)\vec{p}.$$

Hence,

$$\tau = \kappa_y \cos\alpha. \tag{7.23}$$

Replacing κ_y with $\frac{\kappa}{\sin\alpha}$ by Equation 7.21 in Equation 7.23, we get

$$\tau = \left(\frac{\kappa}{\sin\alpha}\right)\cos\alpha,$$

or

$$\frac{\tau}{\kappa} = \frac{\cos\alpha}{\sin\alpha},$$

which is constant and nonzero, since $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. Therefore $\vec{\underline{f}}(s)$ is a general helix for any $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$.

Remark Since $\tau = \kappa_y \cos\alpha$, we notice the following. If $\alpha \in (0, \frac{\pi}{2})$, the general helix is right-handed; if $\alpha \in (\frac{\pi}{2}, \pi)$, the helix is left-handed.

7.5 Summary of Conversion Process

To convert from the general helix to associated plane curve

1. Start with a general helix C, where C is a $C^{(r)}$ curve, with $r \geq 3$, $\kappa \neq 0$, and $\tau \neq 0$. Let C have an allowable representation of $\underline{\mathrm{f}}(s)$.

2. If α is not obvious, calculate important expressions for α by the following method.

   (a) Calculate κ and τ for the general helix.

   (b) Let $c = \frac{\tau}{\kappa}$, where c is known to be constant.

   (c) Let $\cos\alpha = \frac{c}{\sqrt{1+c^2}}$ and $\sin\alpha = \frac{1}{\sqrt{1+c^2}}$. We note that the vector $\vec{u}$ corresponding the fixed direction in space for the general helix is given by $\vec{u} = (\cos\alpha)\,\vec{t} + (\sin\alpha)\,\vec{b}$.

3. Define the plane curve $\vec{y} = \vec{\underline{\mathrm{f}}}_y(s)$, by letting

$$\vec{\underline{\mathrm{f}}}_y(s) = \frac{\vec{\underline{\mathrm{f}}}(s) - \vec{\underline{\mathrm{f}}}(s_0) - (s - s_0)(\cos\alpha)\vec{u}}{\sin\alpha}.$$

4. Use the following equations to get information about the plane curve.

$$\vec{t}_y = (\sin\alpha)\,\vec{t} - (\cos\alpha)\,\vec{b}.$$

$$\vec{p}_y = \vec{p}.$$

$$\vec{b}_y = \vec{u} = (\cos\alpha)\,\vec{t} + (\sin\alpha)\,\vec{b}.$$

$$\kappa_y = \frac{\kappa}{\sin\alpha}.$$

$$\tau_y = 0.$$

Remark If the representation of the general helix is given in terms of a general parameter t, the only change is the definition of the plane curve. In that case, we have $\vec{y} = \vec{\underline{\mathrm{f}}}_y(t)$, defined by letting

$$\vec{\underline{\mathrm{f}}}_y(t) = \frac{\vec{\underline{\mathrm{f}}}(t) - \vec{\underline{\mathrm{f}}}(t_0) - (\underline{\mathrm{s}}(t) - \underline{\mathrm{s}}(t_0))(\cos\alpha)\vec{u}}{\sin\alpha}.$$

where $\underline{\mathrm{s}}(t)$ is given by Definition 3.2.1.

To convert from the associated plane curve to the general helix

1. Start with a $C^{(r)}$ plane curve, with $r \geq 3$ and $\kappa_y \neq 0$. Let the plane curve have an allowable representation of $\vec{\underline{f}}_y(s)$, where $\vec{\underline{f}}_y(s_0) = 0$ so that the plane curve passes through the origin.

2. If $\vec{b}_y$ is not obvious (i.e. $\widehat{k}$ direction), calculate it, where it is known to be constant. Let $\vec{u} = \vec{b}_y$.

3. Let α be an arbitrary angle in the interval $(0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. It is known that choosing $\alpha \in (0, \frac{\pi}{2})$ will produce a right-handed general helix, while choosing $\alpha \in (\frac{\pi}{2}, \pi)$ will produce a left-handed helix.

4. Define the general helix $\vec{x} = \vec{\underline{f}}(s)$ by letting
$$\vec{\underline{f}}(s) = \vec{\underline{f}}(s_0) + (\sin\alpha)\,\vec{\underline{f}}_y(s) + (s - s_0)(\cos\alpha)\,\vec{u},$$
where $\vec{\underline{f}}(s_0)$ is an arbitrary constant vector.

5. Use the following equations to get information about the general helix.
$$\vec{t} = (\sin\alpha)\,\vec{t}_y + (\cos\alpha)\,\vec{b}_y = (\sin\alpha)\,\vec{t}_y + (\cos\alpha)\,\vec{u}.$$
$$\vec{p} = \vec{p}_y.$$
$$\vec{b} = -(\cos\alpha)\,\vec{t}_y + (\sin\alpha)\,\vec{b}_y = (\sin\alpha)\,\vec{t}_y + (\cos\alpha)\,\vec{u}.$$
$$\kappa = (\sin\alpha)\kappa_y.$$
$$\tau = (\cos\alpha)\kappa_y.$$

Remark If the representation of the plane curve is given in terms of a general parameter t, the only change is the definition of the general helix. In that case, we define the general helix $\vec{x} = \vec{\underline{f}}(t)$ by letting
$$\vec{\underline{f}}(t) = \vec{\underline{f}}(t_0) + (\sin\alpha)\,\vec{\underline{f}}_y(t) + (\underline{s}(t) - \underline{s}(t_0))(\cos\alpha)\,\vec{u},$$
where $\vec{\underline{f}}(t_0)$ is an arbitrary constant vector.

7.6 Examples that Illustrate the Conversion Process

We now illustrate the conversion process from Section 7.5 by a number of examples.

Example 7.6.1 *Given the general (cylindrical) helix C, that has an allowable representation $\vec{x} = \vec{\underline{f}}(s) = \left\langle a\cos(\frac{s}{\sqrt{a^2+b^2}}), a\sin(\frac{s}{\sqrt{a^2+b^2}}), (\frac{b}{\sqrt{a^2+b^2}})s \right\rangle$, find the associated plane curve and its properties.*

Solution First we calculate κ and τ for the general helix. We have

$$\frac{d\vec{x}}{ds} = \vec{\underline{f}}'(s) = \left\langle -\left(\frac{a}{\sqrt{a^2+b^2}}\right)\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \left(\frac{a}{\sqrt{a^2+b^2}}\right)\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \left(\frac{b}{\sqrt{a^2+b^2}}\right)\right\rangle. \tag{7.24}$$

Notice that $|\vec{\underline{f}}'(s)| = 1$, so s is indeed arc length parameter, and $\vec{t} = \frac{d\vec{x}}{ds}$. Now by Equations 6.6 and 7.24, we have

$$\kappa\vec{p} = \frac{d\vec{t}}{ds} = \vec{\underline{f}}''(s) = \left\langle -\left(\frac{a}{a^2+b^2}\right)\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\left(\frac{a}{a^2+b^2}\right)\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right\rangle. \tag{7.25}$$

By inspection of Equation 7.25, we see that

$$\vec{p} = \left\langle -\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right\rangle. \tag{7.26}$$

and

$$\kappa = \frac{a}{a^2+b^2}. \tag{7.27}$$

Now $\vec{b} = \vec{t} \times \vec{p}$ so by Equation 7.24 and 7.26, we have

$$\vec{b} = \left\langle \left(\frac{b}{\sqrt{a^2+b^2}}\right)\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\left(\frac{b}{\sqrt{a^2+b^2}}\right)\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \left(\frac{a}{\sqrt{a^2+b^2}}\right)\right\rangle. \tag{7.28}$$

Now by Equations 6.8 and 7.28, we have

$$-\tau\vec{p} = \frac{d\vec{b}}{ds} = \left\langle \left(\frac{b}{a^2+b^2}\right)\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \left(\frac{b}{a^2+b^2}\right)\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right\rangle. \tag{7.29}$$

By inspection of Equation 7.29, we see that

$$\tau = \frac{b}{a^2+b^2}. \tag{7.30}$$

We have verified that we do indeed have a general helix, since

$$\frac{\tau}{\kappa} = \frac{b}{a} = c, \tag{7.31}$$

where c is a constant. Now let

$$\cos\alpha = \frac{c}{\sqrt{1+c^2}} = \frac{b}{\sqrt{a^2+b^2}} \tag{7.32}$$

and

$$\sin\alpha = \frac{1}{\sqrt{1+c^2}} = \frac{a}{\sqrt{a^2+b^2}}. \tag{7.33}$$

Since $\vec{u} = (\cos\alpha)\vec{t} + (\sin\alpha)\vec{b}$, we have

$$\vec{u} = \left(\frac{b}{\sqrt{a^2+b^2}}\right)\left\langle -\left(\frac{a}{\sqrt{a^2+b^2}}\right)\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \left(\frac{a}{\sqrt{a^2+b^2}}\right)\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \left(\frac{b}{\sqrt{a^2+b^2}}\right)\right\rangle$$

$$+\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left\langle \left(\frac{b}{\sqrt{a^2+b^2}}\right)\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\left(\frac{b}{\sqrt{a^2+b^2}}\right)\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \left(\frac{a}{\sqrt{a^2+b^2}}\right)\right\rangle.$$

This simplifies to

$$\vec{u} = \langle 0,0,1\rangle\,. \tag{7.34}$$

We now define the plane curve $\vec{y} = \vec{\underline{f}}_y(s)$, by letting

$$\vec{\underline{f}}_y(s) = \frac{\vec{\underline{f}}(s) - \vec{\underline{f}}(s_0) - (s-s_0)(\cos\alpha)\vec{u}}{\sin\alpha}. \tag{7.35}$$

Let us arbitrarily choose $s_0 = 0$. Substituting Equations 7.32, 7.33, and 7.34 into Equation 7.35, we get

$$\begin{aligned}
\vec{\underline{f}}_y(s) &= \frac{\vec{\underline{f}}(s) - \vec{\underline{f}}(0) - s\left(\frac{b}{\sqrt{a^2+b^2}}\right)\langle 0,0,1\rangle}{\frac{a}{\sqrt{a^2+b^2}}} \\
&= \left(\frac{\sqrt{a^2+b^2}}{a}\right)\vec{\underline{f}}(s) - \left(\frac{\sqrt{a^2+b^2}}{a}\right)\vec{\underline{f}}(0) - s\left(\frac{b}{a}\right)\langle 0,0,1\rangle \\
&= \left(\frac{\sqrt{a^2+b^2}}{a}\right)\vec{\underline{f}}(s) - \left(\frac{\sqrt{a^2+b^2}}{a}\right)\langle a,0,0\rangle + \left\langle 0,0,-\left(\frac{b}{a}\right)s\right\rangle \\
&= \left(\frac{\sqrt{a^2+b^2}}{a}\right)\vec{\underline{f}}(s) + \left\langle -\sqrt{a^2+b^2},0,-\left(\frac{b}{a}\right)s\right\rangle \\
&= \left(\frac{\sqrt{a^2+b^2}}{a}\right)\left\langle a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \left(\frac{b}{\sqrt{a^2+b^2}}\right)s\right\rangle. \\
&\quad + \left\langle -\sqrt{a^2+b^2},0,-\left(\frac{b}{a}\right)s\right\rangle.
\end{aligned}$$

Hence the associated plane curve is given by

$$\vec{y} = \vec{\underline{f}}_y(s) = \left\langle \sqrt{a^2+b^2}\left(\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)-1\right), \sqrt{a^2+b^2}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right\rangle, \tag{7.36}$$

which is a circle $(x-\sqrt{a^2+b^2})^2 + y^2 = a^2+b^2$ in the xy plane.

Now let us derive some of the properties of the associated plane curve. We know that

$$\vec{t}_y = (\sin\alpha)\vec{t} - (\cos\alpha)\vec{b}\,. \tag{7.37}$$

Substituting Equations 7.33 and 7.32 into Equation 7.37, yields

$$\vec{t}_y = \left(\frac{a}{\sqrt{a^2+b^2}}\right)\vec{t} - \left(\frac{b}{\sqrt{a^2+b^2}}\right)\vec{b}. \tag{7.38}$$

Now substituting Equations 7.24 and 7.28 yields, upon simplification,

$$\vec{t}_y = \left\langle -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right\rangle. \tag{7.39}$$

Now $\vec{p}_y = \vec{p}$, so

$$\vec{p}_y = \left\langle -\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right\rangle. \tag{7.40}$$

We have $\vec{b}_y = \vec{u}$, which was found in Equation 7.34, so

$$\vec{b}_y = \langle 0, 0, 1\rangle. \tag{7.41}$$

We have $\kappa_y = \frac{\kappa}{\sin\alpha}$, so by Equations 7.27 and 7.33, we have

$$\kappa_y = \frac{\frac{a}{a^2+b^2}}{\frac{a}{\sqrt{a^2+b^2}}} = \frac{\sqrt{a^2+b^2}}{a^2+b^2} = \frac{1}{\sqrt{a^2+b^2}} \tag{7.42}$$

or

$$\rho_y = \frac{1}{\kappa_y} = \sqrt{a^2+b^2} \tag{7.43}$$

and also we know that

$$\tau_y = 0. \tag{7.44}$$

Example 7.6.2 *Given the general helix C, that has an allowable representation $\vec{x} = \vec{\underline{f}}(t) = \langle t, \frac{\sqrt{2}}{2}t^2, \frac{t^3}{3}\rangle$, find the associated plane curve.*

Solution This Example is similar to Example 7.6.1, except that it differs in that the representation for the curve is not given in terms of the arc length parameter. However, by the remarks in Section 7.5, we see that this is not too much of a difficulty. Furthermore, we already know that we have a general helix, since the curve is a member of class of curves discussed in Theorem 7.2.1. Let us just cite the results of Theorem 7.2.1 for this specific helix.

$$\frac{d\vec{x}}{dt} = \vec{\underline{f}}'(t) = \left\langle 1, \sqrt{2}t, t^2\right\rangle \tag{7.45}$$

$$\frac{d^2\vec{x}}{dt^2} = \vec{\underline{f}}''(t) = \left\langle 0, \sqrt{2}, 2t\right\rangle \tag{7.46}$$

$$\frac{d^3\vec{x}}{dt^3} = \vec{\underline{f}}'''(t) = \langle 0, 0, 2\rangle. \tag{7.47}$$

$$\underline{s}(t) = \frac{t^3}{3} + t. \tag{7.48}$$

$$\kappa = \frac{\sqrt{2}}{(1+t^2)^2}. \tag{7.49}$$

$$\tau = \frac{\sqrt{2}}{(1+t^2)^2}. \tag{7.50}$$

Using Properties T5 and B4 for the Unit Tangent Vector and Unit Binormal Vector, we can find $\vec{t}$ and $\vec{b}$, which will allow us to find $\vec{p}$. Doing so, yields,

$$\vec{t} = \frac{1}{t^2+1}\left\langle 1, t\sqrt{2}, t^2 \right\rangle \tag{7.51}$$

$$\vec{p} = \frac{1}{t^2+1}\left\langle -t\sqrt{2}, 1-t^2, t\sqrt{2} \right\rangle \tag{7.52}$$

$$\vec{b} = \frac{1}{t^2+1}\left\langle t^2, -t\sqrt{2}, 1 \right\rangle \tag{7.53}$$

Now $\vec{u} = (\cos\alpha)\vec{t} + (\sin\alpha)\vec{b}$. From Equations 7.51, 7.53, 7.17, 7.18, where $c = 1$, we have

$$\vec{u} = \frac{1}{\sqrt{2}}\frac{1}{t^2+1}\left\langle 1, t\sqrt{2}, t^2 \right\rangle + \frac{1}{\sqrt{2}}\frac{1}{t^2+1}\left\langle t^2, -t\sqrt{2}, 1 \right\rangle = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle. \tag{7.54}$$

We now define the plane curve $\vec{y} = \vec{\underline{\mathrm{f}}}_y(t)$, by letting

$$\vec{\underline{\mathrm{f}}}_y(t) = \frac{\vec{\underline{\mathrm{f}}}(t) - \vec{\underline{\mathrm{f}}}(t_0) - (\underline{\mathrm{s}}(t) - \underline{\mathrm{s}}(t_0))(\cos\alpha)\vec{u}}{\sin\alpha}. \tag{7.55}$$

With $t_0 = 0$, $\underline{\mathrm{s}}(t_0) = 0$, $\vec{\underline{\mathrm{f}}}(t_0) = 0$, and $\cos\alpha = \sin\alpha = \frac{1}{\sqrt{2}}$, Equation 7.55 becomes

$$\vec{\underline{\mathrm{f}}}_y(t) = \frac{\vec{\underline{\mathrm{f}}}(t) - \underline{\mathrm{s}}(t)(\frac{1}{\sqrt{2}})\vec{u}}{\frac{1}{\sqrt{2}}} = \sqrt{2}\vec{\underline{\mathrm{f}}}(t) - \underline{\mathrm{s}}(t)\vec{u}. \tag{7.56}$$

Now substituting Equations 7.48 and 7.54 into Equation 7.56, along with the definition of $\vec{\underline{\mathrm{f}}}(t)$ yields

$$\vec{\underline{\mathrm{f}}}_y(t) = \sqrt{2}\left\langle t, \frac{\sqrt{2}}{2}t^2, \frac{t^3}{3} \right\rangle - \left(\frac{t^3}{3} + t\right)\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle.$$

Upon simplification, we have

$$\begin{aligned} \vec{\underline{\mathrm{f}}}_y(t) &= \left\langle \frac{3t\sqrt{2} - t^3\sqrt{2}}{6}, t^2, \frac{t^3\sqrt{2} - 3t\sqrt{2}}{6} \right\rangle \\ &= \frac{\sqrt{2}}{6}\left\langle 3t - t^3, 3t\sqrt{2}, t^3 - 3t \right\rangle \end{aligned}$$

or

$$\vec{\underline{\mathrm{f}}}_y(t) = t\left\langle \frac{1}{\sqrt{2}}, 1, -\frac{1}{\sqrt{2}} \right\rangle + \frac{1}{3}t^3 < -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} > . \tag{7.57}$$

Example 7.6.3 *Given the plane curve C, that has an allowable representation* $\overrightarrow{y} = \overrightarrow{\underline{f}}_y(t) = \langle t\sin t + \cos t - 1, -t\cos t + \sin t, 0\rangle$, *find a general helix generated by this plane curve.*

Solution In this example, it is obvious that $\overrightarrow{b}_y = \widehat{k} = \langle 0, 0, 1\rangle$, and let $\overrightarrow{u} = \overrightarrow{b}_y = \widehat{k}$. Since we are using a general parameter t, we need to calculate the arc length function $\underline{s}(t)$. Now

$$\frac{d\overrightarrow{y}}{dt} = \overrightarrow{\underline{f}}'_y(t) = \langle t\cos t, t\sin t, 0\rangle\ . \tag{7.58}$$

Letting $t_0 = 0$ arbitrarily, we can now find the arc length function $\underline{s}(t)$, by Definition 3.2.1. Namely,

$$\begin{aligned}
\underline{s}(t) &= \int_0^t \sqrt{\overrightarrow{\underline{f}}'(t)\cdot\overrightarrow{\underline{f}}'(t)}\,dt \\
&= \int_0^t \sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt \\
&= \int_0^t t\,dt.
\end{aligned}$$

Hence,

$$\underline{s}(t) = \frac{t^2}{2}. \tag{7.59}$$

Let α be any arbitrary angle, where $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. As before, we define the general helix $\overrightarrow{x} = \overrightarrow{\underline{f}}(t)$ by letting

$$\overrightarrow{\underline{f}}(t) = \overrightarrow{\underline{f}}(0) + (\sin\alpha)\,\overrightarrow{\underline{f}}_y(t) + \underline{s}(t)(\cos\alpha)\overrightarrow{u},$$

where $\overrightarrow{\underline{f}}(0)$ is an arbitrary constant vector. Hence,

$$\begin{aligned}
\overrightarrow{\underline{f}}(t) &= \overrightarrow{\underline{f}}(0) + (\sin\alpha)\,\langle t\sin t + \cos t - 1, -t\cos t + \sin t, 0\rangle + \frac{t^2}{2}(\cos\alpha)\,\langle 0,0,1\rangle \\
&= \overrightarrow{\underline{f}}(0) + \left\langle (t\sin t + \cos t - 1)(\sin\alpha), (-t\cos t + \sin t)(\sin\alpha), \frac{t^2}{2}(\cos\alpha)\right\rangle .
\end{aligned}$$

Remark If $\alpha \in (0, \frac{\pi}{2})$, the helix is right-handed; if $\alpha \in (\frac{\pi}{2}, \pi)$, the helix is left-handed. Pick a specific α and $\overrightarrow{\underline{f}}(0)$ to identify a particular general helix.

A plot containing both the plane curve, and an associated general helix for $\alpha = \frac{\pi}{4}$, is shown is Figure 7.6. Note: The general helix was scaled up by factor of $\sin\frac{\pi}{4}$, so that the plane curve will be the actual projection of the helix in the plane, and the general helix will lie on the cylinder defined by the plane curve.

7.7 Solving the Natural Equations of a General Helix

Let us start from a set of natural equations for a general helix

$$\begin{cases} \kappa = \underline{\kappa}(s)(> 0) \\ \tau = c\,\underline{\kappa}(s) \end{cases}$$

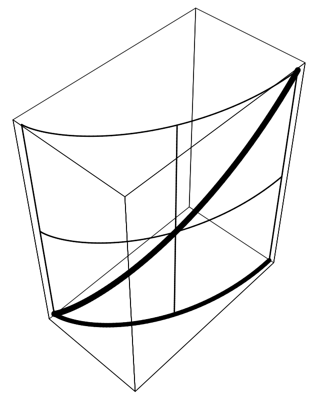


Figure 7.1: General Helix With Associated Plane Curve

where c is a constant, and find the general solution to the natural equations. The method we use is to consider the associated plane curve, because then we may use the method outlined in Section 5.7.

For a general helix, as in Section 7.5, we may identify the angle α, by letting

$$\cos\alpha = \frac{c}{\sqrt{1+c^2}} \tag{7.60}$$

and

$$\sin\alpha = \frac{1}{\sqrt{1+c^2}}. \tag{7.61}$$

From Section 7.5, we have that the curvature κ_y of the associated plane curve is given by

$$\kappa_y = \frac{\kappa}{\sin\alpha}. \tag{7.62}$$

Substituting Equation 7.61 into Equation 7.62 yields

$$\kappa_y = \kappa\sqrt{1+c^2} = \kappa\sqrt{1+\frac{\tau^2}{\kappa^2}} = \kappa\sqrt{\frac{\kappa^2+\tau^2}{\kappa^2}} = \sqrt{\kappa^2+\tau^2}. \tag{7.63}$$

Using Equation 5.17 from Section 5.7, we can find the equation of the plane curve having curvature κ_y. Namely,

$$\vec{\underline{\mathrm{f}}}_y(s) = \left\langle \int_{s_0}^{s} \left[\cos\left(\int_{\gamma_0}^{\gamma} \underline{\kappa}_y(\eta)\,d\eta\right)\right] d\gamma, \int_{s_0}^{s} \left[\sin\left(\int_{\gamma_0}^{\gamma} \underline{\kappa}_y(\eta)\,d\eta\right)\right] d\gamma, 0 \right\rangle. \tag{7.64}$$

Substituting Equation 7.63 into Equation 7.64, we get the equation of the associated plane curve to the general helix:

$$\vec{\underline{\mathrm{f}}}_y(s) = \left\langle \int_{s_0}^{s} \left[\cos\left(\int_{\gamma_0}^{\gamma} \sqrt{(\underline{\kappa}(\eta))^2+(\underline{\tau}(\eta))^2}\,d\eta\right)\right] d\gamma, \int_{s_0}^{s} \left[\sin\left(\int_{\gamma_0}^{\gamma} \sqrt{(\underline{\kappa}(\eta))^2+(\underline{\tau}(\eta))^2}\,d\eta\right)\right] d\gamma, 0 \right\rangle. \tag{7.65}$$

We can now write down the expression for the general helix, since Section 7.5 demonstrated how to find the general helix from the associated plane curve. In particular, the general helix $\vec{x} = \vec{\underline{\mathrm{f}}}(s)$ is given by

$$\vec{\underline{\mathrm{f}}}(s) = \vec{\underline{\mathrm{f}}}(s_0) + (\sin\alpha)\,\vec{\underline{\mathrm{f}}}_y(s) + (s-s_0)(\cos\alpha)\,\vec{u}, \tag{7.66}$$

where $\overrightarrow{\underline{\mathrm{f}}}(s_0)$ is an arbitrary constant vector, and $\overrightarrow{u} = \overrightarrow{b}_y = \widehat{k}$.

Remark When we determined the associated plane curve, we specified its location and orientation. See the remark at the end of Section 5.7. In particular, by Equations 7.20, and 7.21, we have the orientation of the general helix specified by $\overrightarrow{\underline{\mathrm{t}}}(s_0) =< \sin\alpha, 0, \cos\alpha >$ and $\overrightarrow{\underline{\mathrm{p}}}(s_0) =< 0, 1, 0 >$. If a different orientation is desired for the general helix, we write

$$\overrightarrow{\underline{\mathrm{f}}}(s) = \overrightarrow{\underline{\mathrm{f}}}(s_0) + \underset{\sim}{P}\left((\sin\alpha)\overrightarrow{\underline{\mathrm{f}}}_y(s) + (s - s_0)(\cos\alpha)\widehat{k}\right), \tag{7.67}$$

where $\underset{\sim}{P}$ is a rotation matrix.

Chapter 8

$C^{(r)}$ curves, where $r \geq 4$, $\kappa \neq 0$, and $\tau \neq 0$, and General Helices

8.1 A Fourth Derivative Theorem

Theorem 8.1.1 *Let C be a $C^{(r)}$ curve, with $r \geq 4$, $\kappa \neq 0$, and $\tau \neq 0$. For all points $\vec{x} = \vec{\underline{f}}(s)$ on C,*

$$[\vec{\underline{f}}''(s), \vec{\underline{f}}'''(s), \vec{\underline{f}}''''(s)] = \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right).$$

Proof: We have

$$\vec{\underline{f}}'(s) = \frac{d\vec{x}}{ds} = \vec{t}, \tag{8.1}$$

and by Equation 6.6, we have

$$\vec{\underline{f}}''(s) = \frac{d^2\vec{x}}{ds^2} = \frac{d\vec{t}}{ds} = \kappa\vec{p}. \tag{8.2}$$

Now differentiating Equation 8.2, we get

$$\vec{\underline{f}}'''(s) = \frac{d}{ds}\left(\kappa\vec{p}\right) = \frac{d\kappa}{ds}\vec{p} + \kappa\frac{d\vec{p}}{ds}. \tag{8.3}$$

Substituting Equation 6.7 into Equation 8.3, we get

$$\vec{\underline{f}}'''(s) = \frac{d\kappa}{ds}\vec{p} + \kappa(-\kappa\vec{t} + \tau\vec{b}). \tag{8.4}$$

Hence,

$$\vec{\underline{f}}'''(s) = -\kappa^2\vec{t} + \frac{d\kappa}{ds}\vec{p} + \kappa\tau\vec{b}. \tag{8.5}$$

Now differentiating Equation 8.5, we get

$$\vec{\underline{f}}''''(s) = \frac{d}{ds}\left(-\kappa^2\vec{t} + \frac{d\kappa}{ds}\vec{p} + \kappa\tau\vec{b}\right),$$

which when expanded is

$$\vec{\underline{f}}''''(s) = -\frac{d}{ds}(\kappa^2)\vec{t} - \kappa^2\frac{d\vec{t}}{ds} + \frac{d^2\kappa}{ds^2}\vec{p} + \frac{d\kappa}{ds}\frac{d\vec{p}}{ds} + \frac{d\kappa}{ds}\tau\vec{b} + \kappa\frac{d\tau}{ds}\vec{b} + \kappa\tau\frac{d\vec{b}}{ds}. \tag{8.6}$$

Substituting Equations 6.6 through 6.8 into Equation 8.6, yields

$$\vec{\underline{f}}''''(s) = -2\kappa\frac{d\kappa}{ds}\vec{t} - \kappa^3\vec{p} + \frac{d^2\kappa}{ds^2}\vec{p} + \frac{d\kappa}{ds}(-\kappa\vec{t} + \tau\vec{b}) + \frac{d\kappa}{ds}\tau\vec{b} + \kappa\frac{d\tau}{ds}\vec{b} - \kappa\tau^2\vec{p}. \tag{8.7}$$

Hence,

$$\vec{\underline{f}}''''(s) = -3\kappa\frac{d\kappa}{ds}\vec{t} + \left(\frac{d^2\kappa}{ds^2} - \kappa^3 - \kappa\tau^2\right)\vec{p} + \left(2\tau\frac{d\kappa}{ds} + \kappa\frac{d\tau}{ds}\right)\vec{b}. \tag{8.8}$$

Now by Equation 8.2 and 8.5, we have

$$\vec{\underline{f}}''(s) \times \vec{\underline{f}}'''(s) = (\kappa\vec{p}) \times (-\kappa^2\vec{t} + \frac{d\kappa}{ds}\vec{p} + \kappa\tau\vec{b}) = -\kappa^3(\vec{p} \times \vec{t}) + \kappa\frac{d\kappa}{ds}(\vec{p} \times \vec{p}) + \kappa^2\tau(\vec{p} \times \vec{b}). \tag{8.9}$$

Hence,

$$\vec{\underline{f}}''(s) \times \vec{\underline{f}}'''(s) = \kappa^2\tau\vec{t} + \kappa^3\vec{b}. \tag{8.10}$$

Forming the dot product of Equations 8.10 and 8.8, we get

$$\begin{aligned}
[\vec{\underline{f}}''(s), \vec{\underline{f}}'''(s), \vec{\underline{f}}''''(s)] &= (\kappa^2\tau\vec{t} + \kappa^3\vec{b}) \cdot \left[-3\kappa\frac{d\kappa}{ds}\vec{t} + \left(\frac{d^2\kappa}{ds^2} - \kappa^3 - \kappa\tau^2\right)\vec{p} + \left(2\tau\frac{d\kappa}{ds} + \kappa\frac{d\tau}{ds}\right)\vec{b}\right] \\
&= -3\kappa^3\tau\frac{d\kappa}{ds} + \kappa^3\left(2\tau\frac{d\kappa}{ds} + \kappa\frac{d\tau}{ds}\right) \\
&= \kappa^4\frac{d\tau}{ds} - \kappa^3\tau\frac{d\kappa}{ds} \\
&= \kappa^3(\kappa\frac{d\tau}{ds} - \tau\frac{d\kappa}{ds}) \\
&= \kappa^5\left(\frac{\kappa\frac{d\tau}{ds} - \tau\frac{d\kappa}{ds}}{\kappa^2}\right).
\end{aligned}$$

Hence,

$$[\vec{\underline{f}}''(s), \vec{\underline{f}}'''(s), \vec{\underline{f}}''''(s)] = \kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right). \tag{8.11}$$

▮

8.2 A General Helix Test

Theorem 8.2.1 *Let C be a $C^{(r)}$ curve, with $r \geq 4$, $\kappa \neq 0$, and $\tau \neq 0$. C is a general helix if and only if for all points* $\vec{x} = \vec{\underline{\mathrm{f}}}(s)$ on C,

$$[\vec{\underline{\mathrm{f}}}''(s), \vec{\underline{\mathrm{f}}}'''(s), \vec{\underline{\mathrm{f}}}''''(s)] = 0.$$

Proof: Let C be a $C^{(r)}$ curve, with $r \geq 4$, $\kappa \neq 0$, and $\tau \neq 0$. Let C be a general helix. By Lancret's Theorem, $\frac{\tau}{\kappa} = c$, where c is a constant. Hence, by Theorem 8.1.1, we have

$$[\vec{\underline{\mathrm{f}}}''(s), \vec{\underline{\mathrm{f}}}'''(s), \vec{\underline{\mathrm{f}}}''''(s)] = \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right) = \kappa^5 \frac{d}{ds}(c) = 0.$$

Conversely, let C be a $C^{(r)}$ curve, with $r \geq 4$, $\kappa \neq 0$, and $\tau \neq 0$. Also let

$$[\vec{\underline{\mathrm{f}}}''(s), \vec{\underline{\mathrm{f}}}'''(s), \vec{\underline{\mathrm{f}}}''''(s)] = 0.$$

By Theorem 8.1.1, we have

$$[\vec{\underline{\mathrm{f}}}''(s), \vec{\underline{\mathrm{f}}}'''(s), \vec{\underline{\mathrm{f}}}''''(s)] = \kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right),$$

so

$$\kappa^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right) = 0.$$

However, $\kappa \neq 0$, so

$$\frac{d}{ds}\left(\frac{\tau}{\kappa}\right) = 0.$$

Hence, $\frac{\tau}{\kappa}$ is constant. By Lancret's Theorem, C is a general helix. ▮

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Index