I. MATRIX LIE GROUPS

Definition: A matrix Lie group is a closed subgroup $G_M$ of $GL(n, \mathbb{C})$.

Thus if $\{A_m\}_{m=1}^{\infty}$ is any sequence of matrices in $G_M$, and $A_m \to A$ for some $A \in M_n(\mathbb{C})$, then either $A \in G_M$ or $A$ is not invertible.

Example of a Group that is Not a Matrix Lie Group

Let $G_Q^n = \{A \in GL(n, \mathbb{C}) : A = [a_{ij}]_{i,j=1}^{n,n}, \text{ where } a_{ij} \in \mathbb{Q}, \forall i, j\}$.

Then there exists $\{A_m\}_{m=1}^{\infty} \subseteq G_Q^n$ such that $A_m \to \pi \delta \notin G_M$, but $\pi \delta$ is invertible.

Thus $G_Q^n$ is not a matrix Lie group.

Examples of Matrix Lie Groups

$O(n), U(n), Sp(n)$, etc.

Another example is the Heisenberg group $H$ described below:

Let $A = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

Note: If $B, C \in A$, then $BC \in A$

If $B = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, then $B^{-1} = \begin{bmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \in A$.

Then $(A, \cdot)$ is a subgroup of $GL(3, \mathbb{R})$ and if $\{B_m\}_{m=1}^{\infty} \subseteq A$ with $B_m \to B$, then $B \in A$, so $(A, \cdot)$ is a matrix Lie group, called the Heisenberg group $H$. 
Facts about the Heisenberg group $H$

1. $Z(H) = \left\{ \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : b \in \mathbb{R} \right\}$ \hspace{1cm} [Exercise]

2. Let $\mathfrak{h} = \left\{ \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$. Then $\mathfrak{h}$ is the Lie algebra of $H$. \hspace{1cm} [Exercise]

3. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then $\{A, B, C\}$ is a basis for $\mathfrak{h}$.

4. $B^2 = 0$ so $e^{tB} = \delta + tB = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus $Z(H) = \{e^{tB} : t \in \mathbb{R}\}$.

5. $[A, C] = \sim B$ and $[A, B] = [C, B] = \sim 0$ \hspace{1cm} [Exercise]

II. LIE GROUPS

Definition: $G_L$ is a $(C^\infty)$–Lie group if $G_L = (X, T, |A|, \cdot)$ where

1. $(X, T, |A|)$ is a $C^\infty$-manifold

2. $(X, \cdot)$ is a group

3. $\cdot : X \times X \to X$ is a smooth function with respect to the smooth product structure on $X \times X$ and the smooth structure on $X$ (determined by $|A|$)

4. Letting $\varphi_{\cdot x} : X \to X$ be defined by $\varphi_{\cdot x}(z) = z^{-1}$, $\varphi_{\cdot x}$ is a smooth function (with respect to $|A|$)
The Lie Group $G$

Let $X = \mathbb{R} \times \mathbb{R} \times S^1_{\mathbb{C}} = \{(x, y, u) : x, y \in \mathbb{R} \text{ and } u \in S^1_{\mathbb{C}} = \{z \in \mathbb{C} : |z| = 1\}\}$. 

Let $\mathcal{T}_{S^1_{\mathbb{C}}}$ be the subspace topology on $S^1_{\mathbb{C}}$ induced from $\mathbb{C}$.

Let $\mathcal{T}_X$ be the standard product topology on $X$ induced from $\mathcal{T}_{\mathbb{R}}$, the standard metric topology, and $\mathcal{T}_{S^1_{\mathbb{C}}}$. 

Note: $\mathcal{T}_X$ is second countable and Hausdorff.

Let $U_1 = \{e^{i\theta} : \theta_1 \in (0, 2\pi)\}$ and $U_2 = \{e^{i\theta} : \theta_2 \in (-\pi, \pi)\}$.

Let $V_1 = (0, 2\pi)$ and $V_2 = (-\pi, \pi)$.

Let $\varphi_1 : U_1 \rightarrow V_1$ be defined by $z \mapsto \{\text{unique } \theta_1 \in (0, 2\pi) \text{ such that } z = e^{i\theta_1}\}$.

Let $\varphi_2 : U_2 \rightarrow V_2$ be defined by $w \mapsto \{\text{unique } \theta_2 \in (-\pi, \pi) \text{ such that } w = e^{i\theta_2}\}$.

Let $\mathcal{A}_{S^1_{\mathbb{C}}} = \{(U_1, V_1, \varphi_1), (U_2, V_2, \varphi_2)\}$

Then $(S^1_{\mathbb{C}}, \mathcal{T}_{S^1_{\mathbb{C}}}, |\mathcal{A}_{S^1_{\mathbb{C}}}|)$ is a $C^\infty$-manifold. [Exercise]

Via the standard product construction, with product atlas $\mathcal{A}_X$, we have that $(X, \mathcal{T}_X, |\mathcal{A}_X|)$ is a $C^\infty$-manifold. [Exercise]

Note: If $z \in X$, then in each coordinate chart, $z = (x, y, e^{i\theta})$ for an appropriate choice of $\theta$.

Define: $\cdot : X \times X \rightarrow X$ locally (i.e. in each coordinate chart) as 

$$\left((x_1, y_1, u_1) \cdot (x_2, y_2, u_2) = (x_1 + x_2, y_1 + y_2, e^{ix_1y_2}u_1u_2)\right)$$

This is well-defined on $X \times X$ (independent of choice of coordinate chart) [Exercise]

Facts about $\cdot$

1. $\cdot$ is associative [Exercise]

2. $(0,0,1)$ is the identity element

3. $(x,y,u)^{-1} = (-x,-y, e^{iy}u^{-1})$

Thus $(X, \cdot)$ is a group.
Now, $\omega^0$ are smooth in each coordinate chart (by inspection), so are smooth.

Hence $G = (X, T_X, |AX|, \cdot)$ is a Lie group.

### III. EVERY MATRIX LIE GROUP IS A LIE GROUP

Every matrix Lie group $G_M$ is a smooth embedded submanifold of $M_n(\mathbb{C})$ and hence a Lie group.

**Idea of Proof:**

For each point in $G_M$, take “small enough” neighborhood on which $e^{\omega}_g$ is defined to map neighborhood to Lie algebra $\mathfrak{g}$, which is a vector subspace of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$.

To do this formally, we need some facts.

**Definition:**

$$||X||_2 = \left( \sum_{k=1}^{n} \sum_{i=1}^{n} |X_{ki}|^2 \right)^{\frac{1}{2}} \text{ for } X \in M_n(\mathbb{C})$$

**Proposition 1:** If $X \in M_n(\mathbb{C})$ with $||X|| < \varepsilon_n 2$, then $\omega_{g}(e^{\sim}X)$ is defined and $\omega_{g}(e^{\sim}X) = X$.

[Exercise: See Theorem 2.7 in “Lie Groups, Lie Algebras, and Representations”, by Brian Hall]

**Definition:** Let $\varepsilon \in (0, \varepsilon_n 2)$. Then let $U_\varepsilon = \{X \in M_n(\mathbb{C}) : ||X|| < \varepsilon\}$ and $V_\varepsilon = \omega_{g}(U_\varepsilon)$

**Note:** By Proposition 1, $V_\varepsilon$ is open in $M_n(\mathbb{C})$

**Proposition 2:** Suppose $G_M \subseteq GL(n, \mathbb{C})$ is a matrix Lie group with Lie algebra $\mathfrak{g}$. Then there exists $\varepsilon \in (0, \varepsilon_n 2)$ such that for all $A \in V_\varepsilon$, we have $A \in G_M \Leftrightarrow \omega_{g} A \in \mathfrak{g}$.

[Exercise: See Theorem 2.27 in “Lie Groups, Lie Algebras, and Representations”, by Brian Hall]

**Proposition 3:** Every matrix Lie group $G_M$ is a smooth embedded submanifold of $M_n(\mathbb{C})$ and hence a Lie group

**Proof:**

Let $\varepsilon \in (0, \varepsilon_n 2)$.

Let $\mathcal{T}_{G_M}$ be the subspace topology on $G_M$. 
Let $\sim A_0 \in G_M$.

Let $\sim C = 0$.

Then $\sim A_0 = \sim A_0 \delta = \sim A_0 \exp(\sim C) \in \sim A_0 \exp(\sim U_\varepsilon) = \sim A_0 V_\varepsilon$.

Since $V_\varepsilon$ is open in $M_n(\mathbb{C})$ and multiplication by $\sim A_0$ is a homeomorphism onto $\sim A_0 V_\varepsilon$, $\sim A_0 V_\varepsilon$ is open in $M_n(\mathbb{C})$.

Thus $\sim A_0 V_\varepsilon$ is an open neighborhood of $\sim A_0$.

**Note:** $X \in \sim A_0 V_\varepsilon \iff \sim A_0^{-1} X \in V_\varepsilon$, and by Proposition 2, $\sim A_0^{-1} X \in V_\varepsilon \iff \sim \exp(\sim A_0^{-1} X) \in \mathfrak{g}$

Then define $\varphi_{\sim A_0} : \sim A_0 V_\varepsilon \to \mathfrak{g}$ by $\varphi_{\sim A_0}(\sim X) = \sim \exp(\sim A_0^{-1} \sim X)$.

Then, by Proposition 2, $\varphi$ is a well-defined homeomorphism. [Exercise]

Now $\mathfrak{g} \subseteq M_n(\mathbb{C})$, and $\mathfrak{g}$ is a vector space, so $\mathfrak{g}$ is a vector subspace of $M_n(\mathbb{C})$.

Let $\{v_1, \ldots, v_k\}$ be a basis for $\mathfrak{g}$.

Extend $\{v_1, \ldots, v_k\}$ to a basis $\{v_1, \ldots, v_{n^2}\}$ for $M_n(\mathbb{C})$.

Let $\eta : M_n(\mathbb{C}) \to \mathbb{R}^{2n^2}$ be defined by

$$\eta \left( \sum_{i=1}^{n^2} a_i v_i \right) = \left( \Re a_1, \Im a_1, \Re a_2, \Im a_2, \ldots, \Re a_n, \Im a_n \right)$$

Then $\eta$ is a linear isomorphism. [Exercise]

Furthermore, $\eta_{\mathfrak{g}}^{\eta_{\mathfrak{g}}} : \mathfrak{g} \to \mathbb{R}^{2k} \times \{0\}^{2n^2-2k}$ is a linear isomorphism. [Exercise]

Let $\Phi_{\sim A_0} = \eta_{\mathfrak{g}}^{\eta_{\mathfrak{g}}} \circ \varphi_{\sim A_0}$.

Then $\Phi_{\sim A_0}(\sim A_0 V_\varepsilon) = \mathbb{R}^{2k} \times \{0\}^{2n^2-2k} \subseteq \mathbb{R}^{2n^2}$, so is a smooth embedded submanifold of $M_n(\mathbb{C})$.

Then $(\sim A_0 V_\varepsilon, \mathbb{R}^{2k}, \pi_{2k} \circ \Phi_{\sim A_0})$ is a chart for $\sim A_0$.  

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Let \( A = \{(BV_\varepsilon, \mathbb{R}^{2k}, \pi_{2k} \circ \Phi_B) : B \in G_M\}\).

Then \((G_M, T_{G_M}, |A|, \cdot)\), where \(\cdot\) is standard matrix multiplication, is a Lie group.

**IV. NOT EVERY LIE GROUP IS A MATRIX LIE GROUP**

In fact, we will show even more.

Namely, not every Lie group is **algebraically isomorphic** to a matrix Lie group!

**Nilpotent Matrix Lemma**

**Definition:** A matrix \( X \in M_n(\mathbb{R}) \) is called **nilpotent** if there exists \( k \in \mathbb{N} \) such that \( X^k = 0 \).

**Lemma:** If \( X \in M_n(\mathbb{R}) \) is a nonzero nilpotent matrix, then for all nonzero real numbers \( t, e^{tx} \neq 0 \).

**Proof:**

Let \( X \neq 0 \) be a nilpotent matrix, and suppose, by way of contradiction, there exists \( t_0 \in \mathbb{R} \) such that \( t_0 \neq 0 \) and \( e^{t_0X} = 0 \).

Since \( X \) is nilpotent, there exists \( k \in \mathbb{N} \) such that \( X^k = 0 \).

Let \( t \in \mathbb{R} \).

Then

\[
e^{tx} = \delta + tX + \frac{(tX)^2}{2!} + \frac{(tX)^k}{k!} + \frac{(tX)^{k+1}}{(k+1)!} + \cdots
\]

\[
= \delta + \frac{t^2X^2}{2!} + \cdots + \frac{t^{k-1}X^{k-1}}{(k-1)!}
\]

Let \( c_{ij} \) be the \( ij \)th entry of \( X^l \).

Then \( (e^{tx})_{ij} = \delta_{ij} + c_{ij}t + \frac{c_{ij}^2}{2!}t^2 + \cdots + \frac{c_{ij}^{k-1}}{(k-1)!}t^{k-1} \).

Hence there exists polynomials \( \rho_{ij}(t) \) such that \( (e^{tx})_{ij} = \rho_{ij}(t) \). \((1)\)
Now let \( m \in \mathbb{N} \).

Then \( (e^{mX})^m = \delta^m \), so \( e^{mX} = \delta \).

Then \( (e^{mX})_{ij} = \rho_{ij}(mt_0) \), so \( \rho_{ij}(mt_0) = \delta_{ij} \).

Let \( \rho_{ij}(t) = \rho_{ij}(t) - \delta_{ij} \).

Now, if by way of contradiction, \( \rho_{ij}(t) \) is nonconstant, then \( \rho_{ij}(t) \) is nonconstant and has roots \( mt_0 \), for all \( m \in \mathbb{N} \).

Thus \( \rho_{ij}(t) \) has infinitely many roots, violating the Fundamental “\( N \)-th root” Theorem of Algebra.

Thus \( \rho_{ij}(t) \) is constant, i.e. there exists \( c_{ij} \in \mathbb{R} \) such that \( \rho_{ij}(t) = c_{ij} \).

Thus, by (1), for all \( t \in \mathbb{R} \), \( (e^{tX})_{ij} = c_{ij} \).

Letting \( C = [c_{ij}]_{i,j=1}^n \), we have \( e^{tX} = C \).

Then \( \frac{d}{dt}(e^{tX}) = 0 \), so \( e^{tX} = 0 \).

Since this holds for all \( t \in \mathbb{R} \), it holds for \( t = 0 \), so \( e^0X = 0 \).

Thus \( \delta X = 0 \), so \( X = 0 \), a contradiction.

**Relate the Heisenberg matrix Lie group \( H \) to the Lie group \( G \)**

Define \( \Phi : H \to G \) by \[
\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mapsto (a, c, e^{ib})
\]

Then \( \Phi \) is a surjective homomorphism. \([\text{Exercise}]\)

Then \( \ker \Phi = \left\{ \begin{bmatrix} 1 & 0 & 2\pi n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\} = \{ e^{2\pi n h} : n \in \mathbb{Z} \} \quad \text{[Exercise]}\)

Let \( N = \ker \Phi \).

By the 1st Isomorphism Theorem, \( \frac{H}{N} \cong G \).
Definition: A Lie group homomorphism from Lie group \( G_L = (X_1, \mathcal{T}_1, |A_1|, \cdot, 1) \) to Lie group \( H_L = (X_2, \mathcal{T}_2, |A_2|, \cdot, 2) \) if the following two conditions are met:

1. \( \Phi : (X_1, \mathcal{T}_1, |A_1|) \to (X_2, \mathcal{T}_2, |A_2|) \) is smooth
2. \( \Phi : (X_1, \cdot, 1) \to (X_2, \cdot, 2) \) is a homomorphism.

Then we write \( \Phi : G_L \to H_L \).

Definition: Let \( G_L \) be a Lie group. Let \( \mathfrak{g} \) be any (unrelated) Lie algebra. Then a finite-dimensional complex representation of \( G_L \) (resp. \( \mathfrak{g} \)) is a Lie group homomorphism \( \Pi : G \to GL(n, \mathbb{C}) \) (resp. Lie algebra homomorphism \( \pi : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C}) \)).

If \( \Pi \) (resp. \( \pi \)) is injective, then we say that \( \Pi \) (resp. \( \pi \)) is faithful.

Theorem: Let \( \Sigma \) be any finite dimensional representation of \( H \). If \( N \subseteq \mathfrak{k}_\Sigma \), then \( Z(H) \subseteq \mathfrak{k}_\Sigma \).

Proof:

Let \( \Sigma \) be any finite dimensional representation of \( H \).

Let \( \sigma : \mathfrak{h} \to \mathfrak{gl}(n, \mathbb{R}) \) be defined by \( \sigma(X) = \frac{d}{d\lambda} \left[ \Sigma(e^{\lambda X}) \right]_{\lambda=0} \).

Then \( \sigma \) is a finite dimensional representation of \( \mathfrak{h} \), and \( \Sigma(e^{X}) = e^{\sigma(X)} \). [Exercise]

Since \( \sigma \) is Lie algebra homomorphism,

\[
[\sigma(A), \sigma(B)] = \sigma([A, B]) \text{ and } [\sigma(A), \sigma(B)] = [\sigma(C), \sigma(B)] = 0.
\]

Let \( \tilde{F} = \sigma(B) \).

Let \( \{\lambda_1, \ldots, \lambda_n\} \) be the eigenvalues for \( \tilde{F} \) and \( F \) the associated linear operator.

Let \( V_{\lambda_i} = \{ v \in \mathbb{C}^n : (\tilde{F} - \lambda_i) v = 0 \text{ for some } k \} \) (generalized eigenspace).

Let \( v \in V_{\lambda_i} \) for some \( i \in \{1, \ldots, n\} \).

Then \( (F - \lambda_i) F v = F(F - \lambda_i) F v = F v = 0 \), so \( F v \in V_{\lambda_i} \).

Hence \( V_{\lambda_i} \) is invariant under \( F \).
Let \( F_{\lambda_i} = F|_{V_{\lambda_i}} \).

**Note:** \( F_{\lambda_i} - \lambda_i \delta \) is nilpotent.

Now let \( \lambda \) be an eigenvalue of \( F = \sigma(B) \).

Since \( \sigma(A) \) and \( \sigma(C) \) commute with \( \sigma(B) \), they also leave \( V_\lambda \) invariant. \([\text{Exercise}]\)

Now \( \varepsilon_\lambda(\sigma(B)|_{V_\lambda}) = \varepsilon_\lambda([\sigma(A)|_{V_\lambda}, \sigma(C)|_{V_\lambda}]) = 0 \), since the trace of a commutator is zero.

However, \( \varepsilon_\lambda(\sigma(B)|_{V_\lambda}) = \varepsilon_\lambda(\lambda \delta|_{V_\lambda}) = \lambda \varepsilon_\lambda(\delta|_{V_\lambda}) = \lambda \delta|_{V_\lambda} \).

Thus \( \delta|_{V_\lambda} = 0 \).

Now \( \delta|_{V_\lambda} \neq 0 \), since \( \lambda \) is an eigenvalue, so \( \lambda = 0 \).

Hence, for all \( i \), \( F_{\lambda_i} - \lambda_i \delta = F_{\lambda_i} \).

Thus \( F_{\lambda_i} \) is nilpotent for each \( i \).

**Fact:** \( \mathbb{C}^n = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_m} \) \([\text{Exercise}]\)

Thus \( F = \sigma(B) \) is nilpotent.

Now \( N = \{ e^{2\pi nB} : n \in \mathbb{Z} \} \subseteq \kappa \sigma(B) \Sigma \) (hypothesis), so for all \( n \in \mathbb{Z} \), \( \Sigma(e^{2\pi nB}) = \delta \).

Hence, \( e^{2\pi \sigma(B)} = \delta \) for all \( n \in \mathbb{Z} \). \((1)\)

If, by way of contradiction, \( \sigma(B) \neq 0 \), then \( \sigma(B) \) is a nonzero nilpotent matrix, so by the Nilpotent Matrix Lemma, \( e^{t \sigma(B)} \neq \delta \) for all \( t \in \mathbb{R} \) with \( t \neq 0 \), which contradicts \((1)\).

Thus \( \sigma(B) = 0 \).

Now let \( X \in Z(H) \).

Then there exists \( t \in \mathbb{R} \) such that \( X = e^{tB} \).

Then \( \Sigma(X) = \Sigma(e^{tB}) = e^{t \sigma(B)} = e^{t \cdot 0} = \delta \).
Thus \( X \in \mathfrak{z}(\Sigma) \), so \( Z(H) \subseteq \mathfrak{z}(\Sigma) \).

**Proposition:** The Lie group \( G \) has no faithful finite dimensional representations.

**Proof**

Suppose \( \Psi : G \to GL(n, \mathbb{C}) \) is a finite dimensional representation of \( G \).

Then let \( \Sigma = \Psi \circ \Phi : H \to GL(n, \mathbb{C}) \).

Then \( \Sigma \) is a finite dimensional representation of \( H \).

Let \( \widetilde{X} \in N = \mathfrak{k}_{\text{cent}} \Phi \). Then \( \Phi(\widetilde{X}) = \widetilde{\delta} \).

Then \( \Sigma(\widetilde{X}) = \Psi(\Phi(\widetilde{X})) = \Psi(\widetilde{\delta}) = \widetilde{\delta} \), so \( \widetilde{X} \in \mathfrak{k}_{\text{cent}} \Sigma \).

Thus \( N \subseteq \mathfrak{k}_{\text{cent}} \Sigma \).

Then, by the above Theorem, \( Z(H) \subseteq \mathfrak{k}_{\text{cent}} \Sigma \).

Since \( Z(H) \) is nontrivial, \( \mathfrak{k}_{\text{cent}} \Sigma \) is nontrivial, so \( \Sigma \) is not injective.

Thus \( G \) has no faithful finite dimensional representation.

**Now we show that \( G \) is not isomorphic to any matrix Lie group:**

Assume, by way of contradiction, that there exists an isomorphism \( \eta : G \to G_M \) for some matrix Lie group \( G_M \). Then \( \psi_{\text{cent}} : G_M \to GL(n, \mathbb{C}) \) is an injective Lie group homomorphism.

Hence \( \psi_{\text{cent}} \circ \eta : G \to GL(n, \mathbb{C}) \) is an injective Lie group homomorphism, so \( \psi_{\text{cent}} \circ \eta \) is a faithful finite dimensional representation, which is a contradiction to the above proposition.

Thus \( G \) is not isomorphic to any matrix Lie group.

**Note:** Since \( G \cong H \) and since \( H \) is a matrix Lie group, we see that matrix Lie groups are not preserved by taking quotients.