Green's Theorem, Cauchy's Theorem, Cauchy's Formula

These notes supplement the discussion of real line integrals and Green's Theorem presented in §1.6 of our text, and they discuss applications to Cauchy's Theorem and Cauchy's Formula (§2.3).

1. Real line integrals. Our standing hypotheses are that $\gamma : [a,b] \to \mathbb{R}^2$ is a piecewise smooth curve in $\mathbb{R}^2$, and both u and v are real-valued functions defined on an open subset of $\mathbb{R}^2$ that contains $\gamma([a,b])$. We'll suppose further that u and v have continuous first partial derivatives on this open set.

Let's write $x(t) = \operatorname{Re}\gamma(t)$ and $y(t) = \operatorname{Im}\gamma(t)$, so γ is described by the (real) parametric equations:

$$x = x(t),\ y = y(t), \quad t \in [a,b].$$

Example 1. The unit circle, which we've been describing by the complex equation $\gamma(t) = e^{it}, t \in [0,2\pi]$ is, equivalently described by real parametric equations

$$x = \operatorname{Re} e^{it} = \cos t,\ y = \operatorname{Im} e^{it} = \sin t, \quad t \in [0,2\pi].$$

Definition. $$\int_\gamma u\,dx + v\,dy \stackrel{\text{def}}{=} \int_{t=a}^{b} [u(x(t),y(t))x'(t) + v(x(t),y(t))y'(t)]\,dt.$$

In other words, just as with complex line integrals, you just substitute the parameterization of the curve into the symbols in the left-hand integral to define an ordinary Riemann integral on the right.

Example 2. Let γ be the quarter of the unit circle in the first quadrant, from 1 to $(1+i)/\sqrt{2}$. Compute $\int_\gamma y\,dx + x\,dy$.

Solution. Parameterize γ, say by

$$x = \cos t,\ y = \sin t, \quad 0 \le t \le \pi/4.$$

Thus $dx = -\sin t\,dt$ and $dy = \cos t\,dt$, so

$$\begin{aligned}
\int_\gamma y\,dx + x\,dy &\stackrel{\text{def}}{=} \int_{t=0}^{\pi/4} (-\sin^2 t + \cos^2 t)\,dt \\
&= \int_{t=0}^{\pi/4} \cos 2t\,dt = \frac{1}{2}\sin 2t\Big|_0^{\pi/4} \\
&= \frac{1}{2}[\sin\frac{\pi}{2} - \sin 0] \\
&= \frac{1}{2}
\end{aligned}$$

Physical interpretation. Recall from Calculus III that if $\mathbf{F} = u\mathbf{i} + v\mathbf{j}$ is a force field defined on γ, and we write $d\mathbf{R} = dx\mathbf{i} + dy\mathbf{j}$, then $\mathbf{F} \cdot d\mathbf{R}$ can be though of as the work done in by the force field in pushing a particle over γ through a displacement $d\mathbf{R}$, and

$$\int_\gamma u\,dx + v\,dy = \int_\gamma \mathbf{F} \cdot d\mathbf{R}$$

is then the work done by $\mathbf{F}$ in pushing a particle over all of γ.

2. Real vs. Complex line integrals. The complex line integrals we studied in §1.6 can be expressed in terms of the real ones discussed above. Here's how:

Suppose γ is a piecewise smooth curve in $\mathbb{C}$ and f is a complex-valued function that is continuous on an open set that contains γ. Suppose further that f has continuous first partial derivatives on this open set.

Write $f = u + iv$ where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, so both u and v are real-valued functions that are continuous and have continuous first partials on some open set containing γ. Now proceeding formally (meaning: without trying to make sense out of what we're doing), we have: $z = x + iy$, so $dz = dx + idy$, hence

$$f(z)\,dz = (u + iv)(dx + idy) = u\,dx - v\,dy + i(v\,dx + u\,dy)$$

so

$$\int_\gamma f(z)\,dz = \int_\gamma (u + iv)(dx + idy) = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy \tag{1}$$

Proof of (1). γ has a "complex parameterization" $z = \gamma(t) = x(t) + iy(t)$, $t \in [a, b]$, for which the corresponding real parameterization is

$$x = x(t),\;\; y = y(t), \quad t \in [a, b].$$

Now just sort through definitions:

$$\begin{aligned}
\int_\gamma f(z)\,dz &\overset{\text{def}}{=} \int_{t=a}^{b} f(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b [u(x(t), y(t)) + iv(x(t), y(t))][x'(t) + iy'(t)]\,dt \\
&= \int_a^b [u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]\,dt \\
&\qquad + i\int_a^b [v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)]\,dt \\
&\overset{\text{def}}{=} \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy. \qquad \square
\end{aligned}$$

3. Green's Theorem. Here's the version you learned in Calc III:
Suppose Ω is a domain in $\mathbb{R}^2$ whose positively oriented boundary[1] Γ is a finite collection of pairwise disjoint[2] piecewise continuous simple closed curves. Suppose P and Q are continuous functions defined on a larger open set, which contains both Ω and Γ, and suppose P and Q have continuous first partial derivatives on this larger open set. <u>Then</u>:

$$(2)\quad \int_\Gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Amusing application. Suppose Ω and Γ are as in the statement of Green's Theorem. Set $P(x,y) \equiv 0$ and $Q(x,y) = x$. Then according to Green's Theorem:

$$\int_\Gamma x\,dy = \iint_\Omega 1\,dx\,dy = \text{area of } \Omega.$$

Exercise 1. Find some other formulas for the area of Ω. For example, set $Q \equiv 0$ and $P(x,y) = -y$. Can you find one where neither P nor Q is $\equiv 0$?

Serious application. *Suppose Ω and Γ are as in the statement of Green's Theorem: Ω a bounded domain in the plane and Γ it's positively oriented boundary (a finite union of simple, pairwise disjoint, piecewise continuous closed curves). Suppose f is a complex-valued function that is analytic on an open set that contains both Ω and Γ.*

<u>Then</u>

$$\int_\Gamma f(z)\,dz = 0.$$

Proof. We'll use the *real* Green's Theorem stated above. For this write f in real and imaginary parts, $f = u + iv$, and use the result of §2 on each of the curves that makes up the boundary of Ω. The result is:

$$\int_\Gamma f(z)\,dz = \underbrace{\int_\Gamma u\,dx - v\,dy}_{I} + i\underbrace{\int_\Gamma v\,dx + u\,dy}_{II}.$$

By Green's Theorem,

$$I = -\iint_\Omega \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy \quad \text{and} \quad II = \iint_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy.$$

Since f is analytic on Ω, both integrands on the right sides of these equations vanish identically on Ω, hence the integrals are zero. This completes the proof. $\square$

Example 3. Suppose γ is the unit circle and p a complex number of modulus > 1.

Then: $\displaystyle\int_\gamma \frac{dz}{z-p} = 0.$

Proof. Apply the result above with $f(z) = 1/(z-p)$, $\Omega = \{z : |z| < 1\}$, $\Gamma = \gamma$. Then f is analytic on a disc slightly bigger than the unit disc that doesn't contain the point p, say in the disc of radius $(1+|p|)/2$, so the hypotheses of the above result are satisfied, hence so is the conclusion. $\square$

4. The Cauchy Integral Theorem. *Suppose D is a plane domain and f a complex-valued function that is analytic on D (with f' continuous on D). Suppose γ is a simple closed curve in D whose inside*[3] *lies entirely in D. Then:* $\displaystyle\int_\gamma f(z)\,dz = 0.$

Proof. Apply the "serious application" of Green's Theorem to the special case $\Omega =$ the inside of γ, $\Gamma = \gamma$, taking the open set containing Ω and Γ to be D. $\square$

The Cauchy Integral Formula *Suppose f is analytic on a domain D (with f' continuous on D), and γ is a simple, closed, piecewise smooth curve whose whose inside also lies in D. Then for every point p inside of γ:*

$$f(p) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-p}\,dz.$$

Proof. Fix p lying inside γ, and let ε be any positive number small enough so that the disc $\Delta_\epsilon \stackrel{\text{def}}{=} \{z : |z-p| < \varepsilon\}$ lies entirely inside of γ.[4] Let γ_ε be the positively oriented boundary of Δ_ε. Let Ω_ε be the domain that lies between γ and γ_ε.

Note that the positively oriented boundary Γ_ε of Ω_ε is $\gamma - \gamma_\varepsilon$, and that the function

$$g(z) \stackrel{\text{def}}{=} \frac{f(z)}{z-p} \qquad (\zeta \in D\backslash\{p\})$$

is analytic on $D\backslash\{p\}$. Now apply the "serious application" of Green's Theorem proved in the last section to g, with $D\backslash\{p\}$ playing the role of "the open set containing Ω and Γ." The result is:

$$0 = \int_\Gamma g(z)\,dz = \int_{\gamma-\gamma_\varepsilon} g(z)\,dz = \int_\gamma g(z)\,dz - \int_{\gamma_\varepsilon} g(z)\,dz,$$

[3] Recall the *Jordan Curve Theorem* (pp. 56-57): If γ is a simple closed curve in the plane, then the complement of γ consists of two disjoint open sets, one of which, called the *outside* of γ, is unbounded, while the other, called the *inside* of γ, is bounded.

so

$$\int_\gamma g(z)\,dz = \int_{\gamma_\varepsilon} g(z)\,dz,$$

that is:

$$\int_\gamma \frac{f(z)}{z-p}\,dz = \int_{\gamma_\varepsilon} \frac{f(z)}{z-p}\,dz \tag{3}$$

Let's reduce the right-hand side of (3) to an integral over the real interval $[0, 2\pi]$ by the complex parameterization $z = \gamma_\varepsilon(t) = p + \varepsilon e^{it}$, $0 \le t \le 2\pi$. Then $dz = i\varepsilon e^{it}\,dt$ and $z - p = \varepsilon e^{it}$, so

$$\int_{\gamma_\varepsilon} \frac{f(z)}{z-p}\,dz = \int_0^{2\pi} \frac{f(p+\varepsilon e^{it})}{\varepsilon e^{it}}\; i\varepsilon e^{it}\,dt = i\int_0^{2\pi} f(p+\varepsilon e^{it})\,dt \tag{4}$$

Being differentiable on Ω, f is continuous there. In particular, $f(p+\varepsilon e^{it}) \to f(p)$ as $\varepsilon \to 0$, hence

$$\int_0^{2\pi} f(p+\varepsilon e^{it})\,dt \to \int_0^{2\pi} f(p)\,dt = 2\pi f(p) \tag{5}$$

as $\varepsilon \to 0$.[5]

Now on both sides of (3), take the limit as $\varepsilon \to 0$. The left-hand side does not depend on ε, and on the right we use (4) and (5). The result is:

$$\int_\gamma \frac{f(z)}{z-p}\,dz = 2\pi i\, f(p)$$

as promised. □

Example 4. Let γ be any simple closed curve in the plane, oriented positively, and p a point not on γ. Then:

$$\int_\gamma \frac{1}{z-p}\,dz \;=\; \begin{cases} 2\pi i & \text{if } p \text{ is inside of } \gamma \\ 0 & \text{if } p \text{ is outside of } \gamma \end{cases}$$

Proof. The result for p inside γ is just Cauchy's formula for $f \equiv 1$, while for p outside of γ the function $f(z)/(z-p)$ is an analytic function (of z) on an open set Ω containing both γ and its inside region. Thus the integral is zero by the Cauchy *Theorem.* □

[5]Here we've interchanged the limit, as $\varepsilon \to 0$, with the integral. This requires a separate argument, which we'll skip.

Example 5. Suppose γ is the unit circle, oriented counter-clockwise (i.e., positively). Then applying Cauchy's formula with $f(z) = \sin z$, we get

$$\int_\gamma \frac{\sin z}{z - \pi/4}\, dz = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2},$$

while if we take $f(z) = e^z$, then

$$\int_\gamma \frac{e^z}{z - 1/2}\, dz = e^{1/2}$$

Example 6. Find all the possible values of

$$\int_\gamma \frac{1}{z(z-1)}\, dz$$

as γ ranges over all simple, closed, piecewise smooth curves that do not pass through either of the points 0 or 1.

Solution. There are only four possibilities:

(a) Both 0 and 1 lie outside γ. Then the integrand is analytic in an open set containing γ and its inside, hence the integral is zero, by Cauchy's Theorem.

(b) 0 lies inside γ and 1 lies outside. Then Cauchy's formula can be applied, with $f(z) = 1/(z-1)$, whereupon the integral is $2\pi i f(0) = -2\pi i$.

(c) 1 lies inside γ and 0 lies outside. This time apply Cauchy's theorem with $f(z) = 1/z$. Thus the integral is $2\pi i f(1) = 2\pi i$.

(d) Both 0 and 1 lie inside γ. Then a partial fraction expansion (which could have been used for parts (a)—(c) also) shows:

$$\begin{aligned}
\int_\gamma \frac{1}{z(z-1)}\, dz &= \int_\gamma \left(-\frac{1}{z} + \frac{1}{z-1}\right) dz = -\int_\gamma \frac{1}{z}\, dz + \int_\gamma \frac{1}{z-1}\, dz \\
&= -2\pi i + 2\pi i \qquad \text{(by Example 4)} \\
&= 0.
\end{aligned}$$

Exercise 2. Same as Example 6, except now the integral is: $\displaystyle\int_\gamma \frac{1}{z(z-1)(z-2)}\, dz$, and γ ranges through all simple, closed, piecewise smooth curves missing all of the points 0, 1, or 2.