Additional problems for Section 2.5

The Residue Theorem (cf. Theorem 1, page 154). Let \( \gamma \) be a simple, closed, piecewise smooth, simple closed curve with interior region \( \Omega \). Suppose the function \( f \) is analytic on an open set that contains \( \gamma \) and all but a finite subset \( \{z_1, z_2, \ldots, z_n\} \) of \( \Omega \). Then:

\[
\int_{\gamma} f(z) \, dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f; z_j).
\]

1. Use the Residue Theorem to compute all values of

\[
\int_{\gamma} \frac{z + 2}{z(z - 1)(z - 2)} \, dz
\]

for simple, closed, positively oriented, piecewise smooth curves \( \gamma \) that don’t pass through any of the points 0, 1, or 2.

2. (cf. Exercise #6, page 150). Use the residue theorem to find all possible values of

\[
\int_{\gamma} \frac{1}{e^z + 1} \, dz
\]

where \( \gamma \) is any simple closed positively oriented, piecewise smooth curve that doesn’t pass through any of the zeros of \( e^z + 1 \).

3. Picard’s Theorem states that:

   If \( z_0 \) is an essential singularity of an analytic function \( f \), then given any \( w \in \mathbb{C} \) (possibly with one exception) there is a sequence of points \( z_n \to z_0 \) with \( f(z_n) = w \).

To rephrase the conclusion: the image, under \( f \), of any punctured disc

\[
D_\varepsilon \overset{\text{def}}{=} \{ z : 0 < |z - z_0| < \varepsilon \}
\]

that lies in the domain of \( f \), covers the whole plane infinitely often, with at worst one exception.

In short: \( f \) behaves wildly in any punctured disc about an essential singularity!

Your task: Verify Picard’s Theorem for \( f(z) = e^{1/z} \), and \( z_0 = 0 \). Is there an “exceptional point” that’s not a value of \( f \)? If so, what is it?