EXTRA CREDIT PROBLEMS ON LOGARITHMS

**Problem 1.** For $\alpha$ a fixed real number, define

$$L_\alpha(z) = \ln r + i\theta$$

for $z = re^{i\theta}$ with $r > 0$ and $\alpha < \theta < \alpha + 2\pi$.

(a) Show that $L_\alpha$ is analytic on the domain

$$\Omega_\alpha = \{ z = re^{i\theta} \in \mathbb{C} : \alpha < \theta < \alpha + 2\pi \}$$

and that its derivative at each point $z$ of that domain is $1/z$.

(b) Show that $L_\alpha$ is one-to-one on $\Omega_\alpha$ (meaning that if $z_1$ and $z_2$ are points of $\Omega_\alpha$ for which $L_\alpha(z_1) = L_\alpha(z_2)$, then $z_1 = z_2$).

(c) Determine the image $S_\alpha$ of $\Omega_\alpha$ under $L_\alpha$, and show that the exponential function on $S_\alpha$ is the inverse of $L_\alpha$ (meaning: $\exp(L_\alpha(z)) = z$ for all $z \in \Omega_\alpha$ and $L_\alpha(\exp(z)) = z$ for all $z \in S_\alpha$).

**Problem 2.** Let $\Gamma$ be the union of the following three curves in the plane:

- The closed unit interval $[0, 1]$ of the real line.
- The half circle $z = e^{i\theta}$ for $0 \leq \theta \leq \pi$.
- The half-line $(-\infty, -1]$ on the negative real axis.

(a) Sketch $\Gamma$.

(b) Let $\Omega = \mathbb{C}\setminus\Gamma$. Define a branch $\mathcal{L}$ of the logarithm function by

$$\mathcal{L}(re^{i\theta}) = \ln r + i\theta$$

where $\theta$ is the principal branch of the argument if $r > 1$ and $\theta$ is measured between 0 and $-2\pi$ if $0 < r < 1$. Show that the two definitions give the same result for $z$ on the lower half of the unit circle, and that the resulting function is analytic in $\Omega$. Find the derivative of this function.

(c) Compute $\mathcal{L}(\frac{1}{2}i)$ and $\mathcal{L}(2i)$.

(d) Determine the image under $\mathcal{L}$ of $\Omega$. 