Notes on the complex exponential and sine functions (§1.5)

I. Periodicity of the imaginary exponential. Recall the definition: if $\theta \in \mathbb{R}$ then $e^{i \theta} \overset{def}{=} \cos \theta + i \sin \theta$. Clearly $e^{i(\theta + 2\pi)} = e^{i \theta}$ (because of the $2\pi$ periodicity of the sine and cosine functions of ordinary calculus). It’s also clear—from drawing a picture of $e^{i \theta}$ on the unit circle, say—that $2\pi$ is the “minimal period” of the imaginary exponential, in the sense that if $a \in \mathbb{R}$ has the property that $e^{i(\theta + a)} = e^{i \theta}$ for every $\theta \in \mathbb{R}$ then $a = 2\pi n$ for some integer $n$.

II. Periodicity of complex the exponential. Recall the definition: if $z = x + iy$ where $x, y \in \mathbb{R}$, then $e^z \overset{def}{=} e^x e^{iy} = e^x (\cos y + i \sin y)$.

It’s clear from this definition and the periodicity of the imaginary exponential (§1) that $e^{z+2\pi i} = e^z$, i.e.: “The complex exponential function is periodic with period $2\pi i$."

The first thing we want to show in these notes is that the period $2\pi i$ is “minimal” in the same sense that $2\pi$ is the minimal period for the imaginary exponential (and for the ordinary sine and cosine).

The “Minimal Period Theorem” for the complex exponential. If $\alpha \in \mathbb{C}$ has the property!

\[ e^{z+\alpha} = e^z \quad \text{for all } z \in \mathbb{C}, \]

then $\alpha = 2\pi ni$ for some integer $n$!

Proof. If we set $z = 0$ in (1) we see that $e^\alpha = e^0 = 1$. Now write $\alpha$ in cartesian form: $\alpha = a + ib$ where $a, b \in \mathbb{R}$. Then $1 = e^\alpha = e^a e^{ib}$. Take absolute values on both sides of this last equation to obtain $1 = e^a$, so, (because in this last equation we are dealing with the ordinary exponential of calculus) $a = 0$. Thus $\alpha = ib$, hence our previous equation $e^\alpha = 1$ becomes: $e^{ib} = 1$. It follows from the work of §1 that $b = 2\pi n$ for some integer $n$. Thus $\alpha = 2\pi in$, as promised. \qed

Corollary. $e^\alpha = 1 \iff \alpha = 2\pi ni$ for some integer $n$!

Proof. If $e^\alpha = 1$ then for each $z \in \mathbb{C}$:

$e^{z+\alpha} = e^z e^\alpha = e^z$ so $\alpha$ is a period of the complex exponential, and hence, by the Theorem, is an integer multiple of $2\pi i$. \qed
III. Univalence\(^1\) of the complex exponential. The complex exponential is univalent in any open horizontal strip of width \(2\pi\) or less!!

Remark! Width \(2\pi\) is the best we can hope for by the periodicity of the complex exponential noted in §II above.

**Proof of theorem.** We’ll do a little better, and show that:

\[
e^z = e^w \text{ with } |\text{Im} z - \text{Im} w| < 2\pi,
\]

then \(z = w\)!

Thus, for example, if \(a \in \mathbb{R}\) and \(S\) is the (non-open) strip \(\{z \in \mathbb{C} : a < \text{Im} z \leq a + 2\pi\}\), then the complex exponential is univalent on \(S\). Also, if \(S\) is any open ribbon-shaped region of vertical width \(2\pi\) or less (draw a picture!), then the complex exponential is univalent on \(S\).

So suppose \(z\) and \(w\) are complex numbers that satisfy condition (2). We wish to show \(z = w\). Multiply both sides of \(e^z = e^w\) by \(e^{-w}\) and use the addition law for the complex exponential to get \(e^{z-w} = 1\), whereupon the Corollary to the “Minimal Period Theorem” of §II insures that \(z - w = 2\pi n\) for some integer \(n\). Thus \(\text{Im} z - \text{Im} w = 2\pi n\), hence \(|\text{Im} z - \text{Im} w| = 2\pi|n|\). But our hypothesis is that \(|\text{Im} z - \text{Im} w| < 2\pi\), hence \(n = 0\), whereupon \(z = w\). □.

III. Zeros of the complex sine function. Recall that the complex sine function is defined, for \(z \in \mathbb{C}\), as:

\[
\sin z \overset{\text{def}}{=} \frac{e^{iz} - e^{-iz}}{2i}.
\]

The goal of this section is to show that this extension of the usual sine function of calculus to the complex plane does not add any new zeros.

**Theorem.** \(\sin z = 0 \iff z = n\pi\) for some integer \(n\)!

**Proof.** By trigonometry we know that \(\sin \pi n = 0\) for any integer \(n\), so what’s at stake here is the converse: if \(\sin z = 0\) then \(z = \pi n\) for some integer \(n\).

Well, \(\sin z = 0\) implies that \(e^{iz} = e^{-iz}\), so by multiplying both sides by \(e^{iz}\) and using the addition formula for the complex exponential, we see that \(e^{2iz} = 1\), whereupon, by §I, there’s an integer \(n\) such that \(2z = 2\pi n\), i.e., \(z = n\pi\). □

IV. Periodicity of the complex sine function. The minimal period of the complex sine function is \(2\pi\)!

**Proof.** We know that the complex sine function has period \(2\pi\) (because of the \(2\pi i\) periodicity of the complex exponential). The important assertion here is that if, for some complex number \(\alpha\),

\[
\sin(z + \alpha) = \sin z \text{ for all } z \in \mathbb{C},
\]

\(^1\)Here “univalence” means “one-to-one-ness”. Also, I’ll use “univalent” to mean “one-to-one”.

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then $\alpha$ is an integer multiple of $2\pi$.

So suppose we have (3) for some $\alpha \in \mathbb{C}$. Then upon setting $z = 0$ we see that $\sin \alpha = 0$, hence by §III we know that $\alpha$ is an integer multiple of $\pi$. We wish to show that this integer is even. In any case, we now know $\alpha$ is real. Now set $z = \pi/2$ in (3). Then $\sin(\alpha + \frac{\pi}{2}) = \sin \frac{\pi}{2} = 1$, so (since $\alpha$ is real, hence we’re now operating in the realm of ordinary trigonometry):

$$\alpha + \frac{\pi}{2} = \frac{\pi}{2} + 2\pi n$$

for some integer $n$.

whereupon $\alpha = 2\pi n$, as promised. \hfill \Box

V. Univalence of the complex sine function (cf. page 50). The complex sine is univalent on vertical the strip

$$V \overset{\text{def}}{=} \{ z \in \mathbb{C} : 0 < \text{Re} \, z < \frac{\pi}{2} \}.$$

PROOF. Suppose $z, w \in V$ and $\sin z = \sin w$. 

To show! $z = w$.

Well, from the definition of the complex sine we know that

$$e^{iz} - e^{i(-z)} = e^{iw} - e^{i(-w)},$$

so upon rearranging to get “minus powers” on the same side of the equation:

(4) $$e^{iz} - e^{iw} = e^{i(-z)} - e^{i(-w)} = e^{i(-z)} e^{i(-w)} (e^{iw} - e^{iz}).$$

So either

(5) $$e^{iz} - e^{iw} = 0$$

or we can divide both sides of (4) by $e^{iz} - e^{iw}$ to yield:

(6) $$-1 = e^{i(-z)} e^{i(-w)} = e^{-i(z+w)}.$$

Suppose it’s (5) that’s true. Then $e^{iz} = e^{iw}$ so by a now-familiar argument (using the addition formula for the complex exponential, and the result of §II), $z = w + 2\pi n$ for some integer $n$, i.e. $|\text{Re} \, z - \text{Re} \, w| = 2\pi |n|$. But $|\text{Re} \, z - \text{Re} \, w| < \pi$, so $n = 0$, hence $z = w$.

I claim !!! can’t happen! If it did, then we’d have $e^{i\pi} = -1 = e^{-i(z+w)}$, so by a now-familiar argument, we’d get

$$1 = e^{-i(z+w+\pi)},$$

whereupon the “Corollary” in §I would guarantee that $z + w + \pi = 2\pi n$ for some integer $n$, i.e. that $z + w$ would be an odd (possibly negative) multiple of $\pi$. Thus the same would be true of $\text{Re} \, z + \text{Re} \, w$: it would be an odd multiple of $\pi$. But since $z, w \in V$ we know $0 < \text{Re} \, z + \text{Re} \, w < \pi$, so no such “odd multiple” can exist.

Thus it’s only (5) that can happen, and so our theorem is proved. \hfill \Box