REVIEW PROBLEMS FOR EXAM II

In Problems 1–5, $A$ is a $2 \times 2$ matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1/2$, and corresponding eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Answer questions 1–3 without explicitly finding $A$:

(1) Is $A$ symmetric? Give reasons for your answer.

(2) Consider the dynamical system $\vec{v}(n+1) = A\vec{v}(n)$, where $n = 0, 1, 2, \ldots$.
   
   (a) For the initial vector $\vec{v}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, find a closed-form expression for $\vec{v}(n)$ for any positive integer $n$.

   (b) Find $\lim_{n \to \infty} \vec{v}(n)$, and illustrate with a sketch.

(3) Same question as #2, but now for the initial vector $\vec{v}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(4) Find $A$.

(5) Find $A^{100}$.

(6) Identify the plane conic section whose equation is

$$x^2 + 4xy + y^2 = 1$$

(it’s an ellipse or a hyperbola—which one?). Sketch the curve, showing its axes of symmetry and identifying the points where the curve intersects these axes. If the curve is a hyperbola, sketch the asymptotes, and indicate their slopes.

(7) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by:

$$f(x, y) = x^2 + 3xy + y^2 + \sin(x^2 + y^2).$$

Show that $(0, 0)$ is a critical point of $f$ and determine if it’s has a local maximum, a local minimum, or none of the above.

Date: April 6, 2005.
(8) Find real numbers \(a\), \(b\), and \(c\) so that the matrix
\[
\begin{bmatrix}
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & a \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & b \\
\frac{1}{\sqrt{3}} & 0 & c
\end{bmatrix}
\]
is an orthogonal matrix, and then find the inverse of the resulting matrix.