Suppose \( f \) is a function that is differentiable on an open interval \( I \), and \( a \) is some point of \( I \). Let \( L \) denote the linearization of \( f \) at \( a \), i.e., the function defined on the real line by:

\[
L(x) = f(a) + f'(a)(x - a).
\]

So the graph \( y = L(x) \) is the line tangent to the graph \( y = f(x) \) at the point \((a, f(a))\).

Taylor’s Inequality allows us to estimate how much error we commit when we use \( L(x) \) to approximate \( f(x) \).

Taylor’s Inequality. Suppose, in addition to the hypotheses above, that the second derivative of \( f \) exists at each point of \( I \). For \( x \) in \( I \) suppose we know that \(|f''| \leq M \) at every point of the interval between \( a \) and \( x \). Then,

\[
|f(x) - L(x)| \leq \frac{M}{2}(x - a)^2.
\]

Roughly speaking, Taylor’s Inequality asserts that if \( f \) is twice differentiable on \( I \) then: the error involved in linear approximation at \( a \) is on the order of the square of the distance from \( x \) to \( a \), which is very small for \( x \) near \( a \).

Exercises. Use Taylor’s Inequality to:

1. Do \#11, section 3.10 for \( x \geq 0 \).
2. Do \#11, section 3.10 for \(-0.5 \leq x \leq 0\)
3. Estimate the error in the approximation \( \sin x \approx x \) for \(|x| < 0.1 \).