NONCONVEX LINEAR TOPOLOGIES WITH THE HAHN BANACH EXTENSION PROPERTY

D. A. GREGORY AND J. H. SHAPIRO

ABSTRACT. Let \( (E, E') \) be a dual pair of vector spaces. It is shown that whenever the weak and Mackey topologies on \( E \) are different there is a nonconvex linear topology between them. In particular this provides a large class of nonconvex linear topologies having the Hahn Banach Extension Property.

A linear topology \( T \) on a real or complex vector space \( E \) is said to have the \textit{Hahn Banach Extension Property} (HBEP) if every continuous linear functional on a closed subspace of \( (E, T) \) has a continuous linear extension to the whole space. Every (locally) convex topology has the HBEP, by the Hahn Banach Theorem; and even some nonconvex linear topologies have it [1, §7]. In a discussion of this phenomenon P. C. Shields observed that any linear topology between the weak and Mackey topologies of a dual pair has the HBEP, and asked if such a topology could be nonconvex.

The purpose of this note is to settle Shields' question affirmatively:

\textbf{Theorem 1.} Let \( (E, E') \) be a dual pair of vector spaces with the Mackey topology \( T_\kappa \) on \( E \) not equal to the weak topology \( T_* \). Then there is a nonconvex linear topology between \( T_\kappa \) and \( T_* \).

Note that in addition to the HBEP such a topology has all of the separation properties guaranteed for convex spaces by the Hahn Banach Theorem. However, it cannot be metrizable, for if \( T \) is a nonconvex metrizable topology, then the convex hulls of the \( T \)-neighborhoods of \( 0 \) constitute a base for the Mackey neighborhoods of \( 0 \) [4, Proposition 3], hence \( T \) is strictly stronger than its Mackey topology. It is not known if a linear metric space with the HBEP must be convex, although the result has been proved for complete linear metric spaces with bases [4].

The dual pairs \( (E, E') \) for which \( T_* = T_\kappa \) have the following characterization:

\textbf{Theorem 2.} Let \( (E, E') \) be a dual pair of vector spaces. Then \( T_* = T_\kappa \) if and only if the completion of \( (E, T_\kappa) \) is the algebraic dual of \( E' \).

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In particular, let $S$ be an index set, $\omega(S)$ the space of all scalar valued functions on $S$, and $\phi(S)$ the space of scalar valued functions on $S$ which vanish at all but a finite number of points. Then $\omega(S)$ and $\phi(S)$ have a natural duality, and $T_\ast = T_k$ on $\omega(S)$ [2, Problem 18G]. In fact if $E$ is a weakly dense subspace of $\omega(S)$ (in particular if $E$ contains $\phi(S)$), then $\langle E, \phi(S) \rangle$ is a dual pair, and it follows from Theorem 2 that $T_\ast = T_k$ on $E$. Indeed, every dual pair $\langle E, E' \rangle$ for which $T_\ast = T_k$ can be seen to be of this form by setting $E' = \phi(S)$ where $S$ is an index set for a Hamel basis of $E'$.

We turn to the proofs. Theorem 1 requires three lemmas, the first of which follows from an easy induction argument involving the Hahn Banach Theorem.

**Lemma 1.** If $\langle F, F' \rangle$ is a dual pair of infinite dimensional vector spaces, then there are sequences $(x_n)$ in $F$ and $(y_n)$ in $F'$ such that $\langle x_i, y_j \rangle = \delta_{ij}$ for all $i, j$.

**Lemma 2.** Let $\langle E, E' \rangle$ be a dual pair, and let $(x_n), (y_n)$ be sequences in $E, E'$ respectively such that $\langle x_i, y_j \rangle = \delta_{ij}$ for all $i, j$. If $(y_n)$ is weakly bounded, and $p$ is defined on $E$ by

\[
p(x) = \sum_{n=1}^\infty 2^{-n} |\langle x, y_n \rangle|^{1/2}
\]

then the pseudometric $d(x, y) = p(x - y)$ determines a nonconvex linear topology on $E$.

**Proof.** Since $(y_n)$ is a weakly bounded sequence in $E'$, the convergence of the right-hand side of (1) is assured on $E$. Clearly $d$ determines a linear topology $T_p$ on $E$. Let $U_\epsilon = \{ x \in E : p(x) \leq \epsilon \}$. If $T_p$ is convex, then $U_\epsilon$ contains a convex $T_p$-neighborhood of zero; in particular, it contains the convex hull of $U_\epsilon$ for some $\epsilon > 0$. But

\[w_k = \epsilon^{k+1} w_k\]

belongs to $U_\epsilon$ ($k = 1, 2, \ldots$), and yet

\[p(n^{-1}(w_1 + w_2 + \cdots + w_n)) = en^{1/2},\]

which is larger than 1 for $n$ sufficiently large. Thus $U_\epsilon$ does not contain the convex hull of $U_\epsilon$, and we have a contradiction. $T_p$ is therefore nonconvex.

**Lemma 3.** Let $(E, T)$ be a (not necessarily Hausdorff) topological vector space, and $H$ a subspace whose closure has finite codimension. If the induced topology on $H$ is convex, then $T$ is convex.
Proof. Let $\mathcal{H}$ denote the $T$-closure of $H$. It is easily seen that the $T$-closures of the sets in a neighborhood base of zero in $H$ form a neighborhood base of zero in $\mathcal{H}$, hence the induced topology on $\mathcal{H}$ is convex. Since $\mathcal{H}$ has finite codimension, $E$ is the topological direct sum of $\mathcal{H}$ and a finite dimensional Hausdorff topological vector space [3, Chapter 1, 3.5]. Since the induced topologies on $\mathcal{H}$ and the finite dimensional space are convex, $T$ must also be convex.

Proof of Theorem 1. Since $T_s \neq T_h$, there is a weakly compact absolutely convex subset $A$ of $E'$ which is not contained in the closed convex hull of any finite set of points in $E'$. Thus the linear subspace $F'$ of $E'$ spanned by $A$ has infinite dimension, and it follows from Lemma 1 (with $F = E/(F')^\circ$) that there are sequences $(y_i)$ in $A$ and $(x_i)$ in $E$ such that $\langle x_i, y_j \rangle = \delta_{ij}$ for all $i, j$. Since $A$ is weakly bounded it follows from Lemma 2 that the topology $T_s$ defined on $E$ by (1) is not convex. Moreover $p$ is dominated on $E$ by the $T_s$ seminorm

$$p_s(x) = \sup \{ \langle x, y \rangle : y \in A \}$$

so $T_s$ is weaker than $T_h$.

Let $T$ denote the supremum of $T_s$ and $T_p$. Clearly $T$ is a linear topology between $T_s$ and $T_h$. We claim $T$ is not convex. Suppose otherwise. Then $U_\epsilon$ contains a convex $T$-neighborhood $V$ of 0, which in turn contains a set of the form

$$U_\epsilon \cap \{ x : \langle x, z_i \rangle \leq 1, \quad i = 1, 2, \ldots, n \}$$

for some $\epsilon > 0$ and $z_1, z_2, \ldots, z_n$ in $E'$. Let $H$ be the subspace of $E$ on which all the $z_i$ vanish. Since $U_\epsilon \cap H$ contains $V \cap H$, hence the convex hull of $U_\epsilon \cap H$, it follows that the restriction of $p$ to $H$ determines a convex (not necessarily Hausdorff) topology on $H$. Since $H$ is of finite codimension in $E$, it follows from Lemma 3 that $p$ determines a convex topology on $E$, contradicting Lemma 2. Thus $T$ is not convex, and the proof is complete.

Proof of Theorem 2. Since $E$ separates $E'$ it may be regarded as a weakly dense subspace of the algebraic dual $(E')^\ast$ of $E'$. Thus if $T_s = T_h$ then $(E')^\ast$ is the $T_h$-completion of $E$.

Conversely if $(E')^\ast$ is the $T_h$-completion of $E$, then $T_h$ is the restriction to $E$ of the Mackey topology of the dual pair $(E')^\ast, E'$ [2, §18.9, p. 173]. But it follows from [2, Problem 18G] that the Mackey and weak topologies of this dual pair coincide. Since $T_s$ is the restriction to $E$ of the weak topology of the pair, we have $T_s = T_h$, which completes the proof.
References


Queen’s University, Kingston, Ontario, Canada