NONCONVEX LINEAR TOPOLOGIES WITH THE HAHN BANACH EXTENSION PROPERTY

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ABSTRACT. Let $\langle E, E'\rangle$ be a dual pair of vector spaces. It is shown that whenever the weak and Mackey topologies on E are different there is a nonconvex linear topology between them. In particular this provides a large class of nonconvex linear topologies having the Hahn Banach Extension Property.

A linear topology T on a real or complex vector space E is said to have the *Hahn Banach Extension Property* (HBEP) if every continuous linear functional on a closed subspace of (E, T) has a continuous linear extension to the whole space. Every (locally) convex topology has the HBEP, by the Hahn Banach Theorem; and even some nonconvex linear topologies have it [**1**, §7]. In a discussion of this phenomenon P. C. Shields observed that any linear topology between the weak and Mackey topologies of a dual pair has the HBEP, and asked if such a topology could be nonconvex.

The purpose of this note is to settle Shields' question affirmatively:

THEOREM 1. *Let $\langle E, E'\rangle$ be a dual pair of vector spaces with the Mackey topology T_k on E not equal to the weak topology T_s. Then there is a nonconvex linear topology between T_k and T_s.*

Note that in addition to the HBEP such a topology has all of the separation properties guaranteed for convex spaces by the Hahn Banach Theorem. However, it cannot be metrizable, for if T is a nonconvex metrizable topology, then the convex hulls of the T-neighborhoods of 0 constitute a base for the Mackey neighborhoods of 0 [**4**, Proposition 3], hence T is strictly stronger than its Mackey topology. It is not known if a linear metric space with the HBEP must be convex, although the result has been proved for complete linear metric spaces with bases [**4**].

The dual pairs $\langle E, E'\rangle$ for which $T_s = T_k$ have the following characterization:

THEOREM 2. *Let $\langle E, E'\rangle$ be a dual pair of vector spaces. Then $T_s = T_k$ if and only if the completion of (E, T_k) is the algebraic dual of E'.*

Received by the editors October 22, 1969.

AMS Subject Classifications. Primary 4601.

Key Words and Phrases. Hahn Banach Theorem, Mackey topology, weak topology, nonconvex topology.

In particular, let S be an index set, $\omega(S)$ the space of all scalar valued functions on S, and $\phi(S)$ the space of scalar valued functions on S which vanish at all but a finite number of points. Then $\omega(S)$ and $\phi(S)$ have a natural duality, and $T_s = T_k$ on $\omega(S)$ [2, Problem 18G]. In fact if E is a weakly dense subspace of $\omega(S)$ (in particular if E contains $\phi(S)$), then $\langle E, \phi(S)\rangle$ is a dual pair, and it follows from Theorem 2 that $T_s = T_k$ on E. Indeed, every dual pair $\langle E, E'\rangle$ for which $T_s = T_k$ can be seen to be of this form by setting $E' = \phi(S)$ where S is an index set for a Hamel basis of E'.

We turn to the proofs. Theorem 1 requires three lemmas, the first of which follows from an easy induction argument involving the Hahn Banach Theorem.

LEMMA 1. *If $\langle F, F'\rangle$ is a dual pair of infinite dimensional vector spaces, then there are sequences (x_n) in F and (y_n) in F' such that $\langle x_i, y_j\rangle = \delta_{ij}$ for all i, j.*

LEMMA 2. *Let $\langle E, E'\rangle$ be a dual pair, and let (x_n), (y_n) be sequences in E, E' respectively such that $\langle x_i, y_j\rangle = \delta_{ij}$ for all i, j. If (y_n) is weakly bounded, and p is defined on E by*

$$p(x) = \sum_{n=1}^{\infty} 2^{-n} | \langle x, y_n\rangle |^{1/2} \tag{1}$$

then the pseudometric $d(x, y) = p(x-y)$ determines a nonconvex linear topology on E.

PROOF. Since (y_n) is a weakly bounded sequence in E', the convergence of the right-hand side of (1) is assured on E. Clearly d determines a linear topology T_p on E. Let $U_\epsilon = \{x \in E: p(x) \leqq \epsilon\}$. If T_p is convex, then U_1 contains a convex T_p-neighborhood of zero; in particular, it contains the convex hull of U_ϵ for some $\epsilon > 0$. But

$$w_k = \epsilon^2 4^k x_k$$

belongs to U_ϵ ($k = 1, 2, \cdots$), and yet

$$p(n^{-1}(w_1 + w_2 + \cdots + w_n)) = \epsilon n^{1/2},$$

which is larger than 1 for n sufficiently large. Thus U_1 does not contain the convex hull of U_ϵ, and we have a contradiction. T_p is therefore nonconvex.

LEMMA 3. *Let (E, T) be a (not necessarily Hausdorff) topological vector space, and H a subspace whose closure has finite codimension. If the induced topology on H is convex, then T is convex.*

PROOF. Let $\overline{H}$ denote the T-closure of H. It is easily seen that the T-closures of the sets in a neighborhood base of zero in H form a neighborhood base of zero in $\overline{H}$, hence the induced topology on $\overline{H}$ is convex. Since $\overline{H}$ has finite codimension, E is the topological direct sum of $\overline{H}$ and a finite dimensional Hausdorff topological vector space [3, Chapter 1, 3.5]. Since the induced topologies on $\overline{H}$ and the finite dimensional space are convex, T must also be convex.

PROOF OF THEOREM 1. Since $T_s \neq T_k$, there is a weakly compact absolutely convex subset A of E' which is not contained in the closed convex hull of any finite set of points in E'. Thus the linear subspace F' of E' spanned by A has infinite dimension, and it follows from Lemma 1 (with $F = E/(F')^\circ$) that there are sequences (y_n) in A and (x_n) in E such that $\langle x_i, y_j\rangle = \delta_{ij}$ for all i, j. Since A is weakly bounded it follows from Lemma 2 that the topology T_p defined on E by (1) is not convex. Moreover p is dominated on E by the T_k seminorm

$$p_A(x) = \sup\{ | \langle x, y\rangle | : y \in A\}$$

so T_p is weaker than T_k.

Let T denote the supremum of T_s and T_p. Clearly T is a linear topology between T_s and T_k. We claim T is not convex. Suppose otherwise. Then U_1 contains a convex T-neighborhood V of 0, which in turn contains a set of the form

$$U_\epsilon \cap \{x : | \langle x, z_i\rangle | \leqq 1, \quad i = 1, 2, \cdots, n\}$$

for some $\epsilon > 0$ and $z_1, z_2, \cdots, z_n$ in E'. Let H be the subspace of E on which all the z_i vanish. Since $U_1 \cap H$ contains $V \cap H$, hence the convex hull of $U_\epsilon \cap H$, it follows that the restriction of p to H determines a convex (not necessarily Hausdorff) topology on H. Since H is of finite codimension in E, it follows from Lemma 3 that p determines a convex topology on E, contradicting Lemma 2. Thus T is not convex, and the proof is complete.

PROOF OF THEOREM 2. Since E separates E' it may be regarded as a weakly dense subspace of the algebraic dual $(E')^*$ of E'. Thus if $T_s = T_k$ then $(E')^*$ is the T_k-completion of E.

Conversely if $(E')^*$ is the T_k-completion of E, then T_k is the restriction to E of the Mackey topology of the dual pair $\langle (E')^*, E'\rangle$ [2, §18.9, p. 173]. But it follows from [2, Problem 18G] that the Mackey and weak topologies of this dual pair coincide. Since T_s is the restriction to E of the weak topology of the pair, we have $T_s = T_k$, which completes the proof.

References

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