HARDY SPACES THAT SUPPORT NO COMPACT COMPOSITION OPERATORS

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Abstract. We consider, for G a simply connected domain and $0 < p < \infty$, the Hardy space $\mathcal{H}^p(G)$ formed by fixing a Riemann map τ of the unit disc onto G, and demanding of functions F holomorphic on G that the integrals of $|F|^p$ over the curves $\tau(\{|z| = r\})$ be bounded for $0 < r < 1$. The resulting space is usually *not* the one obtained from the classical Hardy space of the unit disc by conformal mapping. This is reflected in our Main Theorem: *$\mathcal{H}^p(G)$ supports compact composition operators if and only if ∂G has finite one-dimensional Hausdorff measure.* Our work is inspired by an earlier result of Matache [14], who showed that the $\mathcal{H}^p$ spaces of half-planes support no compact composition operators. Our methods provide a lower bound for the essential spectral radius which shows that the same result holds with "compact" replaced by "Riesz". We prove similar results for Bergman spaces, with the Hardy-space condition "∂G has finite Hausdorff 1-measure" replaced by "G has finite area." Finally, we characterize those domains G for which every composition operator on either the Hardy or the Bergman spaces is bounded.

1. Introduction

1.1. **The Hardy Spaces.** Our work takes place on a simply connected domain G that is properly contained in the complex plane. Thus the Riemann Mapping Theorem provides holomorphic mappings that take the open unit disc $\mathbb{U}$ univalently onto G. Let us fix one of these "Riemann maps" and call it τ. For $0 < r < 1$ let Γ_r denote the τ-image of the circle $\{|z| = r\}$. Each Γ_r is thus a smooth Jordan curve in G, and the interiors of these curves exhaust G in a regular fashion.

For $0 < p < \infty$ we define $\mathcal{H}^p(G)$ to be the collection of functions F holomorphic on G such that

$$\sup_{0<r<1} \int_{\Gamma_r} |F(w)|^p \, |dw| < \infty. \tag{1}$$

We call these the *Hardy spaces* of G (although "Hardy-Smirnov spaces" would perhaps be more accurate, see [5, Notes, page 184]). If $G = \mathbb{U}$ and τ is the identity map then our definition of $\mathcal{H}^p(G)$ reduces to that of the classical Hardy space H^p of the unit disc. Upon $\mathcal{H}^p(G)$, which is easily seen to be a vector space, we define a distance-measuring functional $F \to \|F\|_p$ by dividing the supremum on the left-hand side of (1) by 2π and taking the p-th root. The functional $\|\cdot\|_p$ is a norm if $p \geq 1$, while if $0 < p < 1$ then its p-th power is a "p-norm" (subadditive and homogeneous of order p [5, §3.2, page 37]); for convenience we will use the term "norm" for both cases. In case $G = \mathbb{U}$ we obtain the usual H^p-norm. In

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general, $\mathcal{H}^p(G)$ is complete in the metric induced by $\|\cdot\|_p$, in fact it turns out to be isometrically isomorphic to H^p. This follows upon making the change of variable $w = \tau(z)$ in the integral on the left-hand side of (1), from which we obtain:

1.2. **Proposition** (see, e.g., [5, Corollary, page 169]). *Suppose F is holomorphic on G. Then $F \in \mathcal{H}^p(G)$ if and only if $(F \circ \tau)(\tau')^{1/p} \in H^p$. In fact, the map $F \to (F \circ \tau)(\tau')^{1/p}$ is a linear isometry taking $\mathcal{H}^p(G)$ onto H^p.*

It follows quickly from this Proposition that if τ is replaced by another Riemann map then the class of functions $\mathcal{H}^p(G)$ is not changed, and although the norm on the space *is* changed, the new one is equivalent to the old in that each is bounded by a constant multiple of the other.

1.3. **Example: G a half-plane.** The linear-fractional map $\tau(z) = (1+z)/(1-z)$ takes $\mathbb{U}$ univalently onto the right half-plane Π. According to Proposition 1.2, a function F holomorphic on Π is in $\mathcal{H}^p(\Pi)$ if and only if the function $z \to F(\tau(z))/(1-z)^{2/p}$ is in H^p. By [11, Chapter VI, page 118] this identifies $\mathcal{H}^p(\Pi)$ as the Hardy space of the right half-plane most often defined by the condition

$$\sup_{x>0} \int_{-\infty}^{\infty} |F(x+iy)|^p \, dy < \infty$$

(see also [5, Exercise 1, page 197]). Note in particular that the map $F \to F \circ \tau$ takes $\mathcal{H}^p(\Pi)$ into, but *not onto,* H^p.

1.4. **Composition operators on $\mathcal{H}^p(G)$.** Suppose Φ is a function holomorphic on G, with $\Phi(G) \subset G$. Then Φ induces a linear *composition operator* C_Φ on the space $\mathrm{Hol}\,(G)$ of all functions holomorphic on G as follows:

$$C_\Phi F = F \circ \Phi \qquad\qquad (F \in \mathrm{Hol}\,(G)).$$

If G is the unit disc then a classical result of Littlewood asserts that every composition operator is bounded on every Hardy space ([13]; see also [5, Chapter 1], [22, Chapters 1 and 9]). Upon the foundation of Littlewood's Theorem has risen a lively interaction between function theory and operator theory that focuses on understanding how properties of composition operators are reflected in the behavior of their inducing maps. Much of this is detailed in the recent books [3] and [22], and conference proceedings [12].

Once boundedness has been established, the next most natural question one can ask about any composition operator is: "Is it compact?" i.e. "Does it take bounded sets into relatively compact ones?" The issue here is to relate the fashion in which the operator compresses $\mathcal{H}^p(G)$ to the way its inducing function compresses G. If $G = \mathbb{U}$ then $\mathcal{H}^p(G) = H^p$ supports many compact composition operators. Two classes of examples that come immediately to mind are: the operators induced by constant functions, and the ones induced by dilation maps $z \to rz$ for $0 \le r < 1$ (see [22, Chapter 2], for example).

However the phenomenon of compactness for composition operators on H^p is actually quite subtle. For example, maps into polygons inscribed in the unit circle induce compact composition operators, while univalent maps whose images contain discs tangent to the circle do not [22, Chapter 2]. The precise characterization of holomorphic selfmaps of $\mathbb{U}$ that induce compact composition operators on H^p involves asymptotic properties of their distribution of values; see [21] or [22, Chapter 10] for details.

It follows readily from Proposition 1.2 that the map $C_\tau : F \to F \circ \tau$ is an isomorphism of $\mathcal{H}^p(G)$ onto H^p if and only if both τ' and its reciprocal are bounded on $\mathbb{U}$ (see [5, Page 169, Theorem 10.2], for example). In this case $\mathcal{H}^p(G)$ coincides with the "conformally invariant" Hardy space defined on G by demanding only that $F \circ \tau$ belong to H^p, and questions about boundedness and compactness of composition operators on $\mathcal{H}^p(G)$ transfer via C_τ to ones already answered for the classical Hardy spaces (e.g. all are bounded, many are compact).

However when either τ' or its reciprocal is *unbounded*, so that our Hardy classes are different from the conformally invariant ones, then surprises await. Regarding boundedness, we show in §6 that the condition of boundedness for both τ' and its reciprocal actually *characterizes* those domains G for which every composition operator is bounded on $\mathcal{H}^p(G)$. However our primary focus is on the existence of compact composition operators, where we are inspired by this recent result of Valentin Matache [14]:

> *If G is a half-plane then $\mathcal{H}^p(G)$ supports* no *compact composition operators.*

The main result of this paper shows that what really lies behind Matache's result is the Hausdorff measure of ∂G. It shows, in particular, that there are even *bounded* simply connected domains G for which $\mathcal{H}^p(G)$ has no compact composition operators!

1.5. **Main Theorem.** *For a simply connected plane domain $G \neq \mathbb{C}$ and an index $p \in (0, \infty)$, the space $\mathcal{H}^p(G)$ supports compact composition operators if and only if the boundary of G has finite one-dimensional Hausdorff measure.*

Essential to our work is a well-known result which rephrases the Hausdorff measure condition on the boundary as a growth restriction on Riemann maps.

1.6. **Theorem.** *Suppose G is a simply connected domain properly contained in $\mathbb{C}$, and suppose τ is a Riemann map for G. Then ∂G has finite one-dimensional Hausdorff measure if and only if $\tau' \in H^1$.*

For a proof see Pommerenke's book [16, Theorem 10.11, pp. 320–321]. Perhaps better known is the special case of G a Jordan domain, for which the result asserts that ∂G is rectifiable if and only if $\tau' \in H^1$. This result, attributed to Privalov and Smirnov, can be found in books of Pommerenke [15, Lemma 10.7, page 319] and Duren [5, Theorem 3.2, page 44]. The proof of Theorem 1.6 referenced above uses this special case, along with the a clever application of the Carathéodory Kernel Theorem.

Theorem 1.6 makes short work of one implication of our Main Theorem: If ∂G has finite one-dimensional Hausdorff measure, so that $\tau' \in H^1$, then the curves $\tau\{|z| = r\}$ all have length bounded by (2π times) the H^1-norm of τ'. It follows that composition operators induced by *constant* selfmaps Φ of G are bounded on $\mathcal{H}^p(G)$. Being of rank one, such operators are therefore compact.

The issue, then, is to prove the converse, i.e. that if some $\mathcal{H}^p(G)$ supports a compact composition operator, then $\tau' \in H^1$. Our work on this problem begins in the next section, where we show how conformal mapping transforms the study of composition operators on $\mathcal{H}^p(G)$ into that of certain *weighted* composition operators on H^p. We use this point of view to prove that the questions of whether or not a given composition operator on $\mathcal{H}^p(G)$ is bounded or compact do not depend on

$p < \infty$. Thus we may restrict attention to the case $p = 2$, and thereby avail ourselves of the comforts of Hilbert Space.

We prove the Main Theorem in §3, and in two following sections give variants concerning Bergman spaces and Riesz operators. The paper concludes with our characterization of those simply connected domains G for which *every* composition operator is bounded on $\mathcal{H}^p(G)$.

1.7. **Remarks.** (a) *The case* $p = \infty$. The reader may wonder why we do not consider the case of $\mathcal{H}^\infty(G)$, the space of bounded holomorphic functions on G, taken in the supremum norm. Clearly every holomorphic selfmap Φ of G induces a bounded composition operator on $\mathcal{H}^\infty(G)$. Note further that C_τ *always* maps $\mathcal{H}^\infty(G)$ isometrically onto $H^\infty = \mathcal{H}^\infty(\mathbb{U})$, so if $\varphi = \tau^{-1} \circ \Phi \circ \tau$, then C_Φ acting on $\mathcal{H}^\infty(G)$ is isometrically similar to C_φ on H^∞. It is a simple exercise to show that C_φ is compact on H^∞ if and only if $\varphi(\mathbb{U})$ has compact closure in $\mathbb{U}$ (see [22, §2.6, Problem 10], for example), hence the same is true with G in place of $\mathbb{U}$.

(b) *Remark on Carleson measures.* The properties of boundedness and compactness for weighted composition operators are readily restated in terms of "Carleson conditions" on pullback measures arising from the change-of-variable formula of measure theory (see, for example, [2]). This point of view can be useful, for example in proving in certain cases that this boundedness and compactness does not depend on p (see further remarks following Proposition 2.4). However by itself the Carleson-measure point of view seldom serves to relate deeper properties of the operators in question with those of their inducing analytic functions. For example, our Theorem 1.5 shows something non-obvious about Theorem 3.4 of [2]: For certain weights its hypotheses are satisfied for *no* holomorphic selfmap φ of $\mathbb{U}$. It would be of interest to characterize, along the lines of the results in [21], those composition operators which are bounded (respectively, compact) on the Hardy spaces we consider here.

2. Weighted Composition Operators

In this section we make the transition from composition operators on $\mathcal{H}^p(G)$ to weighted composition operators on H^p itself, and use the highly developed function theory of Hardy spaces of the disc to show that, for $0 < p, q < \infty$, a composition operator C_Φ is bounded (respectively, compact) on $\mathcal{H}^p(G)$ if and only if it is bounded (respectively, compact) on $\mathcal{H}^q(G)$.

2.1. **From G to $\mathbb{U}$.** As in Section 1, for a simply connected domain $G \neq \mathbb{C}$ we fix a Riemann map τ of $\mathbb{U}$ onto G. Thus to every holomorphic self-map Φ of G there corresponds such a map φ of $\mathbb{U}$ defined by $\varphi = \tau^{-1} \circ \Phi \circ \tau$. For each index $0 < p < \infty$, the Riemann map τ also gives rise to the isometry of Proposition 1.2 which takes $\mathcal{H}^p(G)$ onto H^p; we suppress the dependence on τ, call this mapping V_p, and allow it to operate on all of $\mathrm{Hol}\,(G)$. More formally:

$$(V_pF)(z) = \tau'(z)^{1/p} F(\tau(z)) \qquad\qquad (F \in \mathrm{Hol}\,(G), z \in \mathbb{U}).$$

The operator V_p allows us to associate each composition operator C_Φ on $\mathrm{Hol}\,(G)$ with an operator $A_{\varphi,p} = V_p C_\Phi {V_p}^{-1}$ on $\mathrm{Hol}\,(\mathbb{U})$, which maps H^p boundedly into itself if and only if C_Φ is bounded on $\mathcal{H}^p(G)$ (in which case V_p establishes an isometric similarity between the two operators). Following through the definitions, one sees quickly that for $f \in H^p$:

$$(2) \qquad (A_{\varphi,p}f)(z) = (Q_\varphi(z))^{1/p}\,(f(\varphi(z)), \quad \text{where} \quad Q_\varphi(z) = \frac{\tau'(z)}{\tau'(\varphi(z))} \quad (z \in \mathbb{U}).$$

We note that, because τ' never vanishes, Q_φ is holomorphic on $\mathbb{U}$, hence $A_{\varphi,p}$ is a *weighted composition operator* on $\mathrm{Hol}\,(\mathbb{U})$.

One way of insuring boundedness for $A_{\varphi,p}$ is to demand that Q_φ be bounded on $\mathbb{U}$, so that $A_{\varphi,p}$ will be the product of two operators that are bounded on H^p: C_φ followed by multiplication by $Q_\varphi^{1/p}$. However $A_{\varphi,p}$ may still be bounded even if Q_φ is not; here is just such an example, where $A_{\varphi,p}$ is not only bounded, it is *compact*.

2.2. **Example: $A_{\varphi,p}$ bounded but Q_φ unbounded.** We consider only the case $p = 2$, after which Theorem 2.7 below will take care of the remaining values of p. We write A_φ for $A_{\varphi,2}$.

Let $\tau(z) = 1 - (1-z)^{1/2}$, so that $\tau(\mathbb{U})$ is a teardrop shaped domain symmetric about the real axis, whose boundary meets the unit circle at the point 1, where it makes an angle of $\pi/4$ radians with the unit interval. Let $G = \tau(\mathbb{U})$. It follows from the elementary inequality

$$|1 - w^{1/2}| < |1 - w| \qquad\qquad (\mathrm{Re}\, w > 0)$$

that $G \subset \mathbb{U}$ (set $w = 1 - z$ in this inequality, where $z \in U$). Let Φ be the restriction of τ to G, so

$$\Phi(G) = \tau(G) = \tau(\tau(\mathbb{U})) \subset \tau(\mathbb{U}) = G,$$

i.e., Φ is a holomorphic selfmap of G. The disc map that corresponds to Φ is

$$\varphi = \tau^{-1} \circ \Phi \circ \tau = \tau^{-1} \circ \tau \circ \tau = \tau.$$

Now $\tau'(z) = (1/2)(1-z)^{-1/2}$, so $Q_\varphi(z) = (1-z)^{-1/4}$, an unbounded function on the unit disc.

We show that, nevertheless, A_φ is compact by showing that it is actually a *Hilbert-Schmidt operator.* For this it is enough to show that $\sum_{n=0}^\infty \|A_\varphi(z^n)\|_2^2 < \infty$, and one checks easily that this is equivalent to:

$$(3) \qquad \int_{\partial\mathbb{U}} \frac{|Q_\varphi|}{1 - |\varphi|^2}\, dm \;<\; \infty,$$

where here (and henceforth) m denotes Lebesgue arc-length measure on $\partial\mathbb{U}$, normalized to have total mass one. Because the boundary of G approaches the point 1 nontangentially,

$$1 - |\varphi(\zeta)| \;\geq\; c|1 - \varphi(\zeta)| \qquad (\zeta \in \partial U),$$

where c is a positive constant independent of ζ. Thus, on the unit circle the integrand on the left-hand side of (3) is bounded above by a constant multiple of

$$\frac{|Q_\varphi(\zeta)|}{|1 - \varphi(\zeta)|} \;=\; \frac{1}{|1 - \zeta|^{3/4}}\ ,$$

so that integral is finite, hence A_φ is a Hilbert-Schmidt operator. □

Is there a connection between compactness for C_φ and for $A_{\varphi,p}$? Consider this class of examples, which motivated us to conjecture our Main Theorem: Suppose

G is unbounded and $\varphi \equiv 0$. Then C_φ is compact, but from Theorem 1.6 we see that $Q_\varphi = \tau'/(\tau' \circ \varphi) = \tau'/\tau'(0) \notin H^1$. Thus for $f \in H^p$,

$$A_{\varphi,p} f = Q_\varphi^{1/p} f(0) \in H^p \iff f(0) = 0,$$

so $A_{\varphi,p}$ is not even bounded on H^p.

The next example shows that there is no implication in the other direction, either.

2.3. **Example: $A_{\varphi,p}$ compact but C_φ not.** Let $\tau(z) = (z+1)^2$, so $G = \tau(\mathbb{U})$ is a "heart-shaped" region, symmetric about the real axis, whose inward-pointing cusp has vertex at the origin. Let $\varphi(z) = (1-z)/2$, so that, as noted in the Introduction, C_φ is not compact on any space H^p. Then $Q_\varphi(z) = 2(z+1)/(3-z)$. Now (choosing once again to work only on H^2) we have $A_\varphi f = Q_\varphi^{1/2} \cdot (f \circ \varphi)$. Suppose $\{f_n\}$ is a sequence in the unit ball of H^2 that converges to zero uniformly on compact subsets of $\mathbb{U}$. By Lemma 2.5 below, to see that A_φ is compact it is enough to show that $\|A_\varphi f_n\|_2 \to 0$. For this, let $\varepsilon > 0$ be given, and write

$$\|A_\varphi f_n\|_2^2 = \int_I + \int_J |Q_\varphi|\, |f_n \circ \varphi|^2 \, dm \tag{4}$$

where $I = \{\zeta \in \partial\mathbb{U} : |1+\zeta| < \varepsilon\}$ and $J = \partial\mathbb{U} \backslash I$. Now $|Q_\varphi| < \varepsilon$ on I, so for each n the first integral on the right-hand side of (4) is bounded above by $\varepsilon\|f_n \circ \varphi\|^2 \leq \varepsilon\|C_\varphi\|^2$. Moreover on J we have $|\varphi(\zeta)|$ bounded above by a constant less than one, so $f_n \circ \varphi \to 0$ uniformly on J, hence the second integral on the right-hand side of (4) is less than ε for all sufficiently large n. Thus $\|A_\varphi f_n\|_2^2 < (1 + \|C_\varphi\|^2)\varepsilon$ for all n sufficiently large, which shows that $\|A_\varphi f_n\|_2 \to 0$ and completes the proof that A_φ is compact. □

In what follows we will always reduce questions about the boundedness and compactness of composition operators on the space $\mathcal{H}^p(G)$ to the corresponding questions about the weighted composition operators $A_{\varphi,p}$ on the Hardy spaces of the disc. We devote the rest of this section to showing that such questions do not depend on the index p.

2.4. **Proposition.** *If a composition operator C_Φ is bounded on $\mathcal{H}^p(G)$ for some $0 < p < \infty$, then it is bounded for all such p.*

Remarks. We will prove the equivalent statement for the operators $A_{\varphi,p}$ on H^p. As we remarked in §1.7, this could be done quickly by quoting known theorems involving Carleson measures; this approach was taken by Matache in [14, Theorem 2] for composition operators on the Hardy spaces of a half-plane. In [2] the Carleson-measure approach was used to study more general weighted composition operators on the classical spaces H^p. However here we opt for a more self-contained treatment in the function-theoretic spirit of [24, Theorem 6.1], where the result was first proved for "unweighted" composition operators on Hardy spaces of the disc.

Proof. Suppose first that $A_{\varphi,p}$ is bounded on H^p, and fix $q \neq p$. Fix $f \in H^q$ of norm one. We have the factorization $f = BF$, where B is a Blaschke product and F belongs to H^q, vanishing nowhere on $\mathbb{U}$. Although $|F| \geq |f|$ on $\mathbb{U}$, it turns out that F also has norm one in H^q. Hence $G = F^{q/p}$ lies in H^p and also has norm one in that space. Thus, understanding that all integrals are extended over the entire unit circle, we have:

$$\begin{aligned} \|A_{\varphi,q}f\|_q^q &= \sup_{0\le r<1}\int |Q_\varphi(r\zeta)|\,|f(\varphi(r\zeta))|^q\,dm(\zeta) \\ &\le \sup_{0\le r<1}\int |Q_\varphi(r\zeta)|\,|F(\varphi(r\zeta))|^q\,dm(\zeta) \\ &= \sup_{0\le r<1}\int |Q_\varphi(r\zeta)|\,|G(\varphi(r\zeta))|^p\,dm(\zeta) \\ &= \|A_{\varphi,p}G\|_p^p \le \|A_{\varphi,p}\|^p \end{aligned}$$

so $A_{\varphi,q}$ is bounded on H^q, as desired. □

The proof also shows that $\|A_{\varphi,q}\|^q \le \|A_{\varphi,p}\|^p$, and since p and q are arbitrary, there is actually equality here; of course this equality transfers to the corresponding composition operators on $\mathcal{H}^p(G)$.

In order to deal with compactness we need two preliminary results, both originally observed for unweighted composition operators on H^p by H. J. Schwartz [20]. The first gives a convenient way to restate the notion of compactness for the operators $A_{\varphi,p}$.

2.5. **Lemma.** *For φ a holomorphic selfmap of $\mathbb{U}$ and $0 < p < \infty$, the following are equivalent:*

(a) *The operator $A_{\varphi,p}$ is compact on H^p.*
(b) *Whenever $\{f_n\}$ is a bounded sequence in H^p that converges to zero uniformly on compact subsets of $\mathbb{U}$, then $\|A_{\varphi,p}f_n\|_p \to 0$.*

The proof proceeds exactly as in the unweighted case, using only the following facts:

(i) An operator is compact if and only if it takes bounded sets into relatively compact ones (the *definition* of compactness).
(ii) Bounded subsets of H^p are normal families [5, §3.2, page 36, Lemma].
(iii) $A_{\varphi,p}$ is continuous when H^p is given the topology of uniform convergence on compact subsets of $\mathbb{U}$ (easily checked).

For the details see, for example, [22, §2.4].

To state the second preliminary result we need to regard each holomorphic selfmap φ of $\mathbb{U}$ as extended, via radial limits, to an almost-everywhere defined function on the unit circle. In fact, this can be done for any function in a Hardy space H^p; the resulting boundary function is non-zero almost everywhere, and belongs to the space $L^p(\partial\mathbb{U})$, where its norm equals the H^p norm of the original "interior" function ([5, Chapter 2], [11, Chapter IV.C]).

2.6. **Proposition.** *If, for some $0 < p < \infty$, the operator $A_{\varphi,p}$ is compact on H^p, then $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$.*

Proof. Suppose that $A_{\varphi,p}$ is bounded on H^p, and that $|\varphi| = 1$ on a subset E of the unit circle having positive Lebesgue measure. We will show that $A_{\varphi,p}$ is not compact.

First note that $Q_\varphi^{1/p} = (\tau'/(\tau'\circ\varphi))^{1/p} = A_{\varphi,p}(1) \in H^p$, hence Q_φ has finite, non-zero radial limits at almost every point of $\partial\mathbb{U}$. By the usual measure-theoretic argument we may also assume that $|Q_\varphi| \ge \delta > 0$ on E. The proof now proceeds just as in the case of unweighted composition operators (see [22, page 32], for example): For each integer $n \geq 0$,

$$\|A_{\varphi,p}(z^n)\|_p^p = \int |Q_\varphi|\,|\varphi|^{np}\,dm \geq \int_E |Q_\varphi|\,|\varphi|^{np}\,dm \geq \delta \int_E |\varphi|^{np}\,dm = \delta\, m(E) > 0,$$

where in the last equality we use the fact that $|\varphi| \equiv 1$ on E. Thus, even though $\{z^n\}$ is a sequence of unit vectors in H^p that converges uniformly on compact subsets of $\mathbb{U}$ to zero, we see that $\|A_{\varphi,p}(z^n)\|_p$ stays bounded away from zero. Hence by Lemma 2.5, $A_{\varphi,p}$ is not a compact operator on H^p. □

2.7. **Theorem.** *If C_Φ is compact on $\mathcal{H}^p(G)$ for some $0 < p < \infty$, then it is compact for all such p.*

Proof. It suffices to prove the corresponding result for the operators $A_{\varphi,p}$ on H^p. With the preliminaries already established, the argument is much the same as the corresponding one given in [24, page 493, Proof of Theorem 6.1] for ordinary composition operators on H^p. However for completeness, we give a somewhat detailed sketch of the essentials.

Suppose $A_{\varphi,p}$ is compact on H^p, and fix $0 < q < \infty$. We wish to show that $A_{\varphi,q}$ is compact on H^q. To this end suppose $\{f_n\}$ is a bounded sequence in H^q that converges to zero uniformly on compact subsets of $\mathbb{U}$. It suffices to show that $\|A_{\varphi,q} f_n\|_q \to 0$. Our proof will achieve this only for a subsequence, and we leave it to the reader to show that this is good enough.

As in the proof of Theorem 2.4 we have, for each n, the factorization $f_n = B_n F_n$, where B_n is a Blaschke product, F_n belongs to H^q and does not vanish anywhere on $\mathbb{U}$, and $\|f_n\|_q = \|F_n\|_q$. As before, $G_n = F_n^{q/p} \in H^p$, and $\|G_n\|_p = \|F_n\|_q$ for each n, so the sequence $\{G_n\}$ is bounded in H^p. As we observed previously, bounded subsets of Hardy spaces are normal families, hence by passing to a subsequence we may assume that $\{G_n\}$ converges uniformly on compact subsets of $\mathbb{U}$ to a holomorphic function G that necessarily belongs to H^p. Because $A_{\varphi,p}$ is a compact operator, this implies that $\|A_{\varphi,p}(G_n - G)\|_p \to 0$, from which it follows that the sequence $\{|A_{\varphi,p} G_n|^p\}$ is uniformly integrable with respect to Lebesgue measure on $\partial\mathbb{U}$ (once again we extend H^p functions to $\partial\mathbb{U}$ via radial limits).

Thus we have, at a.e. point of $\partial\mathbb{U}$,

$$|A_{\varphi,p} G_n|^p = |Q_\varphi|\,|G_n \circ \varphi|^p = |Q_\varphi|\,|F_n \circ \varphi|^q \geq |Q_\varphi|\,|f_n \circ \varphi|^q = |A_{\varphi,q} f_n|^q,$$

which shows that $\{|A_{\varphi,q} f_n|^q\}$ is also uniformly integrable on the unit circle. Because $A_{\varphi,p}$ is compact, Proposition 2.6 guarantees that $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$, so $f_n \circ \varphi \to 0$ a.e. on $\partial\mathbb{U}$, thus the same is true for the uniformly integrable sequence $\{|A_{\varphi,q} f_n|^q\}$. Vitali's Convergence Theorem [18, Chapter 6, Exercise 10(b), page 133] now insures that $\int |A_{\varphi,q} f_n|^q\,dm \to 0$. What we have proved is that if $\{f_n\}$ is any bounded sequence in H^q that converges to zero uniformly on compact sets, then some subsequence of $A_{\varphi,q}$-images converges to zero in the norm of H^q. As pointed out above, this implies the desired compactness of $A_{\varphi,q}$. □

3. Proof of Main Theorem

As we pointed out at the end of Section 1, the nontrivial implication of the Main Theorem asserts that:

> *If for some $0 < p < \infty$ the Hardy space $\mathcal{H}^p(G)$ supports a compact composition operator, then $\tau' \in H^1$.*

By Proposition 2.7 it is enough to consider only the case $p = 2$, which we do for the rest of this section. We abbreviate $A_{\varphi,2}$ simply to A_φ, so for $f \in H^2$,

$$A_\varphi f(z) = Q_\varphi(z)^{1/2}(f(\varphi(z))) \quad \text{where} \quad Q_\varphi(z) = \frac{\tau'(z)}{\tau'(\varphi(z))} \qquad (z \in \mathbb{U}).$$

Now H^2 is the Hilbert space of functions holomorphic on the unit disc that have square summable MacLaurin series coefficients. We will make considerable use of its inner product:

$$\langle f,\, g\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} \qquad (f, g \in H^2) \tag{5}$$

where, for example, $\hat{f}(n)$ denotes the n-th MacLaurin coefficient of f.

Our argument depends on identifying a large supply of test functions that reflect the behavior of the operator A_φ. For this purpose we use the *reproducing kernels* for H^2.

3.1. **Reproducing Kernels.** The *reproducing kernel* for a point $a \in \mathbb{U}$ is the function

$$K_a(z) = \frac{1}{1 - \overline{a}z} \qquad (z \in \mathbb{U}). \tag{6}$$

Since the right-hand side of (6) is a function holomorphic in $\mathbb{C}\backslash\{1/\overline{a}\}$, it's clear that $K_a \in H^2$ for each $a \in \mathbb{U}$. The terminology "reproducing kernel" comes from the formula

$$\langle f,\, K_a\rangle = f(a) \qquad (f \in H^2, a \in \mathbb{U}), \tag{7}$$

which follows immediately upon expanding the right-hand side of (6) in a geometric series, and using the definition (5) of the inner product in H^2. The utility of reproducing kernels for our purposes depends on the following result, where A_φ^* denotes the Hilbert-space adjoint of A_φ.

3.2. **Lemma.** *If the operator A_φ is bounded on H^2, then for each $a \in \mathbb{U}$,*

$$A_\varphi^* K_a = \overline{Q_\varphi(a)}^{1/2} K_{\varphi(a)} \qquad (f \in H^2).$$

Proof. For each $f \in H^2$ we compute:

$$\begin{aligned} \langle f,\, A_\varphi^* K_a\rangle &= \langle A_\varphi f,\, K_a\rangle = (A_\varphi f)(a) = Q_\varphi(a)^{1/2}\ f(\varphi(a)) \\ &= Q_\varphi(a)^{1/2}\ \langle f,\, K_{\varphi(a)}\rangle = \langle f,\, \overline{Q_\varphi(a)}^{1/2} K_{\varphi(a)}\rangle, \end{aligned}$$

which, in view of the arbitrariness of f, yields the desired result. □

If $\tau(z) \equiv z$ then Lemma 3.2 asserts that $C_\varphi^* K_a = K_{\varphi(a)}$, a result which figures importantly in the study of compact composition operators (see, e.g., [22, §3.4, page 43]). If Q_φ were in H^∞, then the result would follow from this and a similarly proved fact about multiplication operators: *If $\psi \in H^\infty$ and M_ψ is the operator on H^2 of multiplication by ψ, then $M_\psi^* K_a = \overline{\psi(a)} K_a$.* Because of Example 2.2, however, Lemma 3.2 required its own proof.

Here is a down payment on the proof of the Main Theorem.

3.3. **Proposition.** *Suppose Φ is a holomorphic selfmap of G that induces a compact composition operator on $\mathcal{H}^2(G)$,* and has a fixed point in G. *Then $\tau' \in H^1$.*

Proof. We are assuming that $\Phi(a) = a$ for some $a \in G$, hence the associated holomorphic selfmap $\varphi = \tau^{-1} \circ \Phi \circ \tau$ of $\mathbb{U}$ fixes the point $b = \tau^{-1}(a)$. Since C_Φ is assumed to be compact on $\mathcal{H}^2(G)$, the (unitarily equivalent) operator A_φ is compact on H^2. Now by Lemma 3.2:

$$A_\varphi^* K_b = \overline{Q_\varphi(b)}\, K_{\varphi(b)} = K_b \tag{8}$$

where in the last equality we use the fact that $\varphi(b) = b$, hence $Q_\varphi(b) = \tau'(b)/\tau'(\varphi(b)) = 1$ and $K_{\varphi(b)} = K_b$.

Thus the complex number 1 is an eigenvalue of A_φ^*, hence belongs to its spectrum, and therefore lies as well in the spectrum of A_φ. Since A_φ is compact the Riesz Theorem (see, for example, [22], pages 95 and 99–101) guarantees that 1 is actually an *eigenvalue* of A_φ, so there exists $f \in H^2\backslash\{0\}$ with $A_\varphi f = f$. Thus $g = f/(\tau')^{1/2}$ is a function holomorphic on $\mathbb{U}$ with $g \circ \varphi = g$. Since φ fixes the point b and is not an automorphism (by the compactness of A_φ and Proposition 2.6), its iterates φ_n tend pointwise to b [22, §5.2, Prop. 1], hence $g(z) = g(\varphi_n(z)) \to g(b)$ as $n \to \infty$ for each $z \in \mathbb{U}$. Since f is not identically zero, this shows that $g = f/(\tau')^{1/2}$ is a non-zero constant, hence $(\tau')^{1/2}$, being a non-zero constant multiple of a function in H^2, also belongs to H^2. Thus $\tau' \in H^1$, as desired. □

We finish the proof of the Main Theorem by showing that whenever C_Φ is compact on $\mathcal{H}^2(G)$ then Φ *must* have a fixed point in G. This result is well known for $G = \mathbb{U}$, where it was first obtained by Caughran and Schwartz, using the Denjoy-Wolff and Julia-Carathéodory Theorems ([7], see also [22, §5.5]). Thus it should come as no surprise that these theorems will also play a crucial role in what is to follow.

3.4. **Angular Derivatives and boundary fixed points.** A classical idea that figures importantly in the study of composition operators is that of *angular derivative.* We know that each holomorphic selfmap φ of $\mathbb{U}$—and indeed *any* function in a Hardy space of the disc—has a (finite) non-tangential limit at almost every point of the unit circle. Suppose $\eta \in \partial\mathbb{U}$ and φ has such a limit $\varphi(\eta)$ at that point. If, in addition, the difference quotient $(\varphi(\eta) - \varphi(z))/(\eta - z)$ has a finite nontangential limit at that point, we write this limit as $\varphi'(\eta)$, and call it the *angular derivative* of φ at η. The existence of the angular derivative at a boundary point expresses a certain conformality for φ at that point.

If, at a point η of $\partial\mathbb{U}$, the map φ has nontangential limit η, then we call η a *boundary fixed point* of φ. If φ has finite angular derivative at such a fixed point, then by conformality, $\varphi'(\eta)$ must be non-negative, and by the Schwarz-Pick Lemma it must be strictly positive (see [22, Chapter 4] for details).

The Denjoy-Wolff Theorem guarantees that if φ has no fixed point in $\mathbb{U}$ then it has a (necessarily unique) boundary fixed point η that attracts all the orbits of φ. As if to reflect this attracting property, this so-called *Denjoy-Wolff point* of φ is determined uniquely among all possible boundary fixed points of φ by the fact that φ has finite angular derivative at η satisfying $\varphi'(\eta) \le 1$ (see [22, Chapter 5] for a detailed exposition of all these matters).

3.5. **The Koebe Distortion Theorem.** This result, the final piece in our puzzle, asserts that for any univalent map τ on $\mathbb{U}$,

$$\tau'(0)\frac{1-|z|}{(1+|z|)^3} \;\le\; |\tau'(z)| \;\le\; \tau'(0)\frac{1+|z|}{(1-|z|)^3} \qquad (z \in \mathbb{U})$$

(see [6, Theorem 2.5, page 32], [16, Theorem 1.3, page 9]). For our purposes these inequalities are best rephrased in terms of the *invariant derivative* of τ, defined by:

$$\delta[\tau](z) = |\tau'(z)|(1-|z|^2) \qquad (z \in \mathbb{U}),$$

and so named because for any conformal automorphism α of $\mathbb{U}$:

$$\delta[\tau\circ\alpha] = \delta[\tau]\circ\alpha, \tag{9}$$

(an immediate consequence of the fact that $|\alpha'(z)| = (1-|\alpha(z)|^2)/(1-|z|^2)$ for each $z \in \mathbb{U}$; see [8, page 3], for example.) When expressed in terms of the invariant derivative, the Koebe Distortion Theorem becomes

$$\left(\frac{1-|z|}{1+|z|}\right)^2 \le \frac{\delta[\tau](z)}{\delta[\tau](0)} \;\le\; \left(\frac{1+|z|}{1-|z|}\right)^2 \qquad (z \in \mathbb{U}). \tag{10}$$

We will require a conformally invariant version of this result. For $w \in \mathbb{U}$ let

$$\alpha_w(z) = \frac{w-z}{1-\overline{w}z} \qquad (z \in \mathbb{U}), \tag{11}$$

so α_w is a conformal automorphism of $\mathbb{U}$ that interchanges w with the origin, and is its own compositional inverse. Apply (10) to $\tau\circ\alpha_w$, replace z by $\alpha_w(z)$, and use the invariance property (9); the result is

$$\left(\frac{1-|\alpha_w(z)|}{1+|\alpha_w(z)|}\right)^2 \le \frac{\delta[\tau](z)}{\delta[\tau](w)} \;\le\; \left(\frac{1+|\alpha_w(z)|}{1-|\alpha_w(z)|}\right)^2 \qquad (z, w \in \mathbb{U}). \tag{12}$$

3.6. **Main Theorem—finale.** Let's review where we stand. We wish to prove Theorem 1.5 which, in view of Theorem 1.6, asserts that $\mathcal{H}^p(G)$ supports compact composition operators if and only if $\tau' \in H^1$. So far we have observed that if $\tau' \in H^1$ then composition operators induced by *constant* self-maps of G are bounded, hence, being of rank one, are compact on $\mathcal{H}^p(G)$ (see remarks following Theorem 1.6). Toward the converse we have shown that if a selfmap Φ of G *with a fixed point in* G induces a compact composition operator on $\mathcal{H}^p(G)$, then $\tau' \in H^1$ (Proposition 3.3). Thus to finish our proof we need only show that if a self-map Φ of G induces a compact composition operator on $\mathcal{H}^p(G)$, then it must have a fixed point in G. We will prove the contrapositive statement:

> *If Φ has no fixed point in G then C_Φ is not compact on any of the spaces $\mathcal{H}^p(G)$.*

By Theorem 2.7 it is enough to prove this for $\mathcal{H}^2(G)$. Equivalently it is enough to show that A_φ is non-compact on H^2, where $\varphi = \tau^{-1}\circ\Phi\circ\tau$. Now φ has no fixed point in $\mathbb{U}$, so by the Denjoy-Wolff Theorem it has a boundary fixed point η at which the angular derivative $\varphi'(\eta)$ exists, with $\varphi'(\eta) \le 1$. Without loss of generality we may assume $\eta = 1$.

We turn once more to our reproducing kernel test functions. For $a \in \mathbb{U}$ let $k_a = K_a/\|K_a\|$, where here, and for the rest of this section, "$\|\cdot\|$" denotes the norm of the space H^2. So $\{k_a : |a| \leq 1\}$ is a family of unit vectors in H^2; we claim that it converges weakly to zero as $|a| \to 1-$, i.e. for every $f \in H^2$,

$$\langle k_a,\, f\rangle = \frac{f(a)}{\|K_a\|} \to 0 \quad \text{as} \quad |a| \to 1-.$$

Note that

$$\|K_a\|^2 = \langle K_a,\, K_a\rangle = K_a(a) = \frac{1}{1-|a|^2}\ , \tag{13}$$

so our weak convergence statement can be rephrased as follows: *For every* $f \in H^2$*,*

$$|f(a)| = \mathrm{o}\left(\frac{1}{(1-|a|^2)^{1/2}}\right) \quad \text{as} \quad |a| \to 1-. \tag{14}$$

This is the "little-oh" version of a well-known growth estimate on functions of class H^2 that results from applying the Cauchy-Schwarz inequality to (7):

$$|f(a)| = |\langle f,\, K_a\rangle| \leq \|f\|\,\|K_a\| = \|f\|\,(1-|a|^2)^{-1/2}. \tag{15}$$

The estimate (14) is trivially true for a dense subset of H^2 (the polynomials, for example), and it is an easy exercise to transfer the result to all of H^2 thanks to the uniformity of the "big-oh" estimate (15).

Now compact operators take weakly convergent sequences into norm convergent ones, and have compact adjoints. Thus if A_φ were compact then the same would be true of its adjoint, hence $A_\varphi^* k_a$ would converge to zero in the norm of H^2 as $|a| \to 1-$. Therefore to prove A_φ non-compact we need only show that this does not happen, i.e., that $\limsup_{|a|\to 1-} \|A_\varphi^* k_a\| > 0$.

Our argument hinges once again on Lemma 3.2, which implies that for $0 \leq r < 1$:

$$\|A_\varphi^* k_r\|^2 = \frac{|\tau'(r)|}{|\tau'(\varphi(r))|}\,\frac{\|K_{\varphi(r)}\|^2}{\|K_r\|^2} \;=\; \frac{|\tau'(r)|(1-r^2)}{|\tau'(\varphi(r))|(1-|\varphi(r)|^2)} \;=\; \frac{\delta[\tau](r)}{\delta[\tau](\varphi(r))}\ , \tag{16}$$

whereupon our invariant version (12) of the Distortion Theorem yields

$$\|A_\varphi^* k_r\| \;\geq\; \frac{1-|\alpha_r(\varphi(r))|}{1+|\alpha_r(\varphi(r))|} \qquad\qquad (0 \leq r < 1), \tag{17}$$

where α_r is defined by (11) (here we also use the fact that $|\alpha_w(z)| = |\alpha_z(w)|$ for all $z, w \in \mathbb{U}$).

Upon doing some arithmetic and recalling that φ has a finite angular derivative at 1, we see that

$$\alpha_r(\varphi(r)) \;=\; \frac{\left(\frac{1-\varphi(r)}{1-r}\right)-1}{r\left(\frac{1-\varphi(r)}{1-r}\right)+1} \;\to\; \frac{\varphi'(1)-1}{\varphi'(1)+1} \quad \text{as} \quad r \to 1-,$$

Since $0 < \varphi'(1) \leq 1$, the limit on the right is non-positive, hence letting $r \to 1-$ in (17) we obtain:

$$\liminf_{r\to 1-} \|A_\varphi^* k_r\| \;\geq\; \frac{1-\left(\frac{1-\varphi'(1)}{1+\varphi'(1)}\right)}{1+\left(\frac{1-\varphi'(1)}{1+\varphi'(1)}\right)} \;=\; \varphi'(1) > 0. \tag{18}$$

This establishes the non-compactness of A_φ^*, hence also that of A_φ, and therefore of C_Φ; it completes the proof of our Main Theorem. □

3.7. **Remark on spectra.** In the proof of Proposition 3.3 we could, without loss of generality, have assumed that the fixed point b of φ is the origin (for example, by taking τ to map the origin to the fixed point of Φ). This would give the matrix of A_φ with respect to the orthonormal basis $\{z^n\}$ a particularly revealing form. Note that the n-th column of this matrix is the sequence of MacLaurin coefficients of $A_\varphi(z^n) = Q_\varphi^{1/2}\varphi^n$. Since $\varphi(0) = 0$ and $Q_\varphi(0) = 1$, for $n > 0$ the MacLaurin expansion in question begins with $\varphi'(0)^n z^n$, and for $n = 0$ it is the single term $Q_\varphi(0) = 1$. Thus the matrix of A_φ with respect to this basis is lower triangular, with 1 in the upper left-hand corner, and $\varphi'(0)^n$ at the n-th position of the diagonal for $n \geq 1$. The adjoint therefore has *upper* triangular matrix with diagonal equal to the sequence of complex conjugates of the original diagonal, showing once again that 1 is an eigenvalue of A_φ^* (with eigenfunction $\equiv 1$), but now additionally that $\overline{\varphi'(0)}^n$ is an eigenvalue for $n \geq 1$. Thus $\varphi'(0)^n$, while not necessarily an eigenvalue of A_φ, at least belongs to its spectrum $\sigma(A_\varphi)$:

If A_φ is bounded on H^2 and $\varphi(0) = 0$ then $\{\varphi'(0)^n\}_1^\infty \cup \{1\} \subset \sigma(A_\varphi)$.

In case A_φ is *compact*, the Riesz Theory insures that every non-zero element of its spectrum is an eigenvalue. An argument similar to the one given in the proof of Proposition 3.3, but now using the uniqueness assertion of Koenigs's Theorem (see [22, §6.1], for example) shows that the only possible eigenvalues of A_φ are the matrix diagonal elements. Thus, just as in the unweighted case (see [7], [22, §6.2]):

If A_φ is compact and $\varphi(0) = 0$, then $\sigma(A_\varphi) = \{\varphi'(0)^n\}_1^\infty \cup \{0, 1\}$.

Transferring this result back to G we obtain:

3.8. **Theorem.** *Suppose Φ is a holomorphic selfmap of G for which C_Φ is compact on $\mathcal{H}^2(G)$. Then Φ fixes a point $a \in G$ and $\sigma(C_\Phi) = \{\Phi'(a)^n : n \geq 1\} \cup \{0, 1\}$.*

This result was originally proved for $G = \mathbb{U}$ by Caughran and Schwartz [7], to whom we owe arguments of §3.7.

4. Bergman Spaces

For G a simply connected domain properly contained in $\mathbb{C}$ and $0 < p < \infty$, the *Bergman space* of G, denoted $L_a^p(G)$, is the subspace of $L^p(G)$ consisting of functions holomorphic on G. Here G is understood to be endowed with Lebesgue area measure dA, normalized so that the unit disc has area 1, hence the norm of $L^p(G)$ is defined by:

$$\|F\|_p = \left\{ \int_G |f|^p \, dA \right\}^{1/p} \qquad (F \in L^p(G)).$$

For holomorphic self-maps Φ of G we can ask, in the Bergman setting, the same questions about the induced composition operators C_Φ that we asked for Hardy spaces. In this section we observe that the Hardy-space methods work almost word for word to give:

4.1. **Theorem.** *Suppose G is a simply connected domain properly contained in the plane, and $0 < p < \infty$. Then $L_a^p(G)$ supports a noncompact composition operator if and only if G has finite area.*

As in the Hardy case, one direction is easy: if G has finite area then $L_a^p(G)$ contains the constant functions, hence the composition operators induced by constant self-maps of G are bounded, and since they have rank one, are compact. For the other direction we proceed as before, reducing the problem to one about weighted composition operators on the Bergman space $L^2_a(\mathbb{U})$.

4.2. **Weighted composition operators on $\boldsymbol{L^p_a}(\mathbb{U})$.** Continuing in the spirit of our previous work, let us fix a Riemann map τ of $\mathbb{U}$ onto G, and for F holomorphic on G write

$$V_pF = (\tau')^{2/p}(F\circ\tau)\,.$$

The change of variable formula for integrals with respect to area measure shows that V_p is an isometry of $L^p_a(G)$ onto $L^p_a(\mathbb{U})$. In particular, since $L^p_a(\mathbb{U})$ is complete in its natural metric [26, Theorem 4.13, page 47], the same is true of $L^p_a(G)$.

Given Φ a holomorphic self-map of G, and $\varphi = \tau^{-1}\circ\Phi\circ\tau$ its disc counterpart, the operator $B_{\varphi,p} = V_pC_\Phi V_p^{-1}$ is linear on $\mathrm{Hol}\,(\mathbb{U})$, and questions of boundedness or compactness of C_Φ on $L^p_a(G)$ are equivalent to the same questions for $B_{\varphi,p}$ on $L^p_a(\mathbb{U})$. One easily derives the concrete representation of $B_{\varphi,p}$:

$$B_{\varphi,p}f = \left(\frac{\tau'}{\tau'\circ\varphi}\right)^{2/p}(f\circ\varphi) \qquad\qquad (f\in \mathrm{Hol}\,(\mathbb{U})).$$

Using this representation and the Bergman space version of the Carleson-measure results mentioned just after the statement of Proposition 2.4, we see again that the questions considered here do not depend on p. (See [26, §6.2], for example; unfortunately, the classically inspired arguments of §2.4 and §2.7 are no longer available in the Bergman case.) For the rest of the argument, then, it is enough to consider only the case $p = 2$. We write B_φ instead of $B_{\varphi,2}$. Our argument depends, as in the Hardy space case, on knowing how the Hilbert-space adjoint of B_φ acts on reproducing kernels. Now the $L^2_a(\mathbb{U})$-reproducing kernel K_a for the point $a\in\mathbb{U}$ is the function

$$K_a(z) = \frac{1}{(1-\overline{a}z)^2} \qquad\qquad (z\in\mathbb{U})$$

(see [26, §4.1]), for which

$$\|K_a\| = \langle K_a,\, K_a\rangle^{1/2} = \sqrt{K_a(a)} = \frac{1}{1-|a|^2} \qquad (a\in\mathbb{U}). \tag{19}$$

Repeating the proof of Lemma 3.2 one finds that for each $a\in\mathbb{U}$:

$$B^*_\varphi K_a = \overline{Q_\varphi(a)}K_{\varphi(a)} \quad \text{where, as previously,} \quad Q_\varphi(a) = \frac{\tau'(a)}{\tau'(\varphi(a))}\,. \tag{20}$$

4.3. **The case where $\boldsymbol{\Phi}$ has a fixed point in $\boldsymbol{G}$.** Suppose C_Φ is compact on $L^2_a(G)$ and Φ fixes a point of G. Then B_φ is compact on $L^2_a(\mathbb{U})$ and φ fixes a point b of $\mathbb{U}$. Just as in the Hardy space situation, it follows from (20) that $B^*_\varphi K_b = K_b$, hence 1 is an eigenvalue of B^*_φ, so once again by compactness it is also an eigenvalue of B_φ, with an eigenfunction $f\in L^2_a(\mathbb{U})$. Proceeding as in the proof of Proposition 3.3 we see that f/τ' is a holomorphic function fixed by C_φ, hence (because $\varphi(b) = b$) it must be a non-zero constant. Since $f\in L^2_a(\mathbb{U})$, the same must be true of τ', and therefore G must have finite area. □

4.4. **Proof of Theorem 4.1, completed.** As in §3 we complete the proof of Theorem 4.1 by showing that if a composition operator on $L^2_a(G)$ is induced by a map having no fixed point in G, then that operator cannot be compact.

As in §3.6, we may assume that the corresponding disc map φ has 1 as a boundary fixed point, with $0 < \varphi'(1) \le 1$. Then upon setting $k_r = K_r/\|K_r\|$, where $0 \le r < 1$, we obtain from (20) that

$$\|B^*_\varphi k_r\| \ = \ \frac{\delta[\tau](r)}{\delta[\tau](\varphi(r))} \tag{21}$$

(the only difference between this and the corresponding Hardy space calculation (16) being that the norm on the left-hand side of (21) is *not* squared). Thus the argument that concluded §3.6 now yields

$$\liminf_{r\to 1-} \|B^*_\varphi k_r\| \ge \varphi'(1)^2 > 0.$$

As in §3.6, the vectors k_r converge weakly to zero in $L^2_a(\mathbb{U})$ as $r \to 1-$, hence B^*_φ, and therefore B_φ, is not compact. This completes the proof of Theorem 4.1. □

4.5. **Weighted Bergman spaces.** For $W : G \to (0,\infty)$ a continuous function, let $L^p_a(G,W)$ denote the collection of holomorphic functions F on G with $|F|^p$ integrable with respect to the measure $W dA$. The resulting space is easily seen to be a closed subspace of $L^p(G, W dA)$, hence complete in the metric induced from that space. Fix a Riemann map τ taking $\mathbb{U}$ onto G, and set $w = W \circ \tau$. The map V_p of §4.2 now furnishes an isometry taking $L^p_a(G,W)$ onto $L^p_a(\mathbb{U},w)$, and the map $B_{\varphi,p} = V_p\, C_\Phi\, V_p^{-1}$ acting on $L^p_a(\mathbb{U},w)$ is still given by the formula of §4.2. Once again, Carleson-measure arguments in the Bergman setting show the boundedness or compactness of these operators to be independent of the index p. Moreover the argument of §4.3 goes through without change to show that: *If Φ has a fixed point in G and C_Φ is compact, then $\int_G W\, dA < \infty$.*

If, in addition, the weight is "standard," i.e., $w(z) = (\alpha+1)(1-|z|^2)^\alpha$ for some $\alpha > -1$, then the argument that settled the unweighted case works almost *verbatim* to show that compact composition operators can only be induced by maps with an interior fixed point. Indeed, suppose Φ has no fixed point in G, so that we may assume, as before, that the corresponding disc map φ has its Denjoy-Wolff point at 1. Then the reproducing kernel in $L^2_a(\mathbb{U},w)$ for the point $a \in \mathbb{U}$ is $K_a(z) = (1-\overline{a}z)^{-(\alpha+2)}$ (see [3, Problem 2.1.5, page 27] or [26, §6.4]), hence the calculation (19) applied to the current situation yields $\|K_a\| = (1-|a|^2)^{-(1+\alpha/2)}$. Upon writing $k_a = K_a/\|K_a\|$ we obtain the following analogue of (21):

$$\|B^*_\varphi k_r\| \ = \ \frac{\delta[\tau](r)}{\delta[\tau](\varphi(r))} \left[\frac{1-r^2}{1-|\varphi(r)|^2}\right]^{\alpha/2}.$$

As before, the first factor on the right has lower limit no less that $\varphi'(1)^2$ as $r \to 1-$. By the Julia-Carathéodory Theorem [22, §5.5], the second factor tends to $\varphi'(1)^{-\alpha/2}$ as $r \to 1-$. Thus

$$\liminf_{r\to 1-} \|B^*_\varphi k_r\| \ge \varphi'(1)^{2-\alpha/2} > 0,$$

so B_φ is not compact. Thus we have shown that:

> *A Bergman space $L^p_a(G,W)$ with "standard" weight W supports a compact composition operator if and only if $\int_G W\, dA < \infty$.*

More generally the same will be true if, whenever φ has a finite angular derivative at 1, then $\|K_{\varphi(r)}\|$ and $\|K_r\|$ are comparable as $r \to 1-$. It might be of interest to explore what happens for weights for which this property fails.

4.6. **Remarks.** (a) The Hardy spaces of the unit disc often play the role of limiting cases of the standard-weighted Bergman spaces of the disc, as the exponent α descends to -1. Thus it might seem strange that the result of setting $\alpha = -1$ in the last inequality of §4.5 does not yield the corresponding estimate of §4.4. The seeming discrepancy results from the fact that for L^2-Bergman spaces, regardless of the weight, the unit-disc realization of C_Φ is $B_\varphi : f \to Q_\varphi \cdot (f \circ \varphi)$, whereas for H^2 it is $A_\varphi : f \to (Q_\varphi)^{1/2} \cdot (f \circ \varphi)$ (here, as always, $Q_\varphi = \tau'/(\tau' \circ \varphi)$).

(b) It might be of interest to see if our results can be generalized to Bergman spaces of multiply connected domains G. Here it seems less natural to attempt to transfer the situation to the unit disc; it might be more productive to develop techniques that work directly on G.

5. Riesz Operators

In 1954 A. F. Ruston [19] introduced for Banach spaces a class of operators containing the compacts, yet spectrally indistinguishable from them (see also [4, Chapter 3]). Of particular importance to us, each such "Riesz operator" has the property—guaranteed for compact operators by the Riesz Theory—that every non-zero point of the spectrum is an eigenvalue. In this section we show that the methods of §3 extend to Riesz operators on $\mathcal{H}^p(G)$. There are two surprises here: First, our duality arguments do not require concrete representation of the dual space, and second, they work equally well for the non-locally convex case $0 < p < 1$. In the interest of clarity we focus most of this section on the more familiar setting $1 \le p < \infty$, relegating the non-locally convex situation to concluding remarks.

To define the class of Riesz operators, let $\mathcal{L}(X)$ denote the collection of bounded operators on a Banach space X, and $\mathcal{K}(X)$ the collection of compact operators. When endowed with the operator norm, $\mathcal{L}(X)$ is a Banach algebra in which $\mathcal{K}(X)$ is a two-sided closed ideal. The work of this section focuses on the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$, commonly known as the *Calkin Algebra*. If $T \in \mathcal{L}(X)$ we denote by $\|T\|_e$ the norm, in the Calkin Algebra, of its coset modulo $\mathcal{K}(X)$, and call this the *essential norm* of T. Thus $\|T\|_e$ is the distance from T to the closed subspace of compact operators on X. Similarly the *essential spectrum* $\sigma_e(T)$ of T is the spectrum of its coset mod $\mathcal{K}(X)$ in the Calkin Algebra.

$T \in \mathcal{L}(X)$ is called a *Riesz operator* if $\sigma_e(T) = \{0\}$. Thus Riesz operators correspond to quasi-nilpotent elements of the Calkin algebra. Thanks to the spectral radius formula for Banach algebras:

T is a Riesz operator if and only if $r_e(T) = \lim_n \|T^n\|_e^{1/n} = 0$.

Compact operators are clearly Riesz, and as we mentioned above, Riesz operators have the property that their spectra are indistinguishable from those of compact operators [4, Chapter 3]. In particular, the continuity of the Fredholm index implies, for Riesz operators, that if λ is a non-zero spectral point, then the operator $T - \lambda I$ has index zero, hence λ is an eigenvalue of finite multiplicity.

Note that there are many non-compact operators that are Riesz; for example any non-compact nilpotent operator has this property. In our composition operator setting the map $\varphi(z) = (1 - z)/2$ induces a non-compact composition operator on H^p, but the square of this operator is compact (see, e.g., [22, §2.6, Exercise 4]). Thus the original operator is Riesz, but not compact. See [1] For a Riesz composition operator, no power of which is compact.

Here is (the Banach-space version of) the main theorem of this section:

5.1. **Theorem.** *Suppose that G is a simply connected domain properly contained in $\mathbb{C}$, and that $1 \le p < \infty$. Then $\mathcal{H}^p(G)$ supports a Riesz composition operator if and only if ∂G has finite one-dimensional Hausdorff measure.*

Our result for Bergman spaces, Theorem 4.1, has a similar "Rieszification." We leave it to the reader to state this result, and to extract its proof from the arguments given below for Hardy spaces.

As in Sections 3 and 4 one direction is easy. We have already seen that finite Hausdorff 1-measure for ∂G implies that $\mathcal{H}^p(G)$ supports composition operators that are compact (the ones induced by constant maps, for example), hence Riesz. For the other direction our strategy remains the same, however in order to have one proof that works for all cases, we forego the urge to represent the dual space $(H^p)^*$ of H^p concretely, and simply treat it as the abstract space of bounded linear functionals on H^p. In this interpretation the reproducing kernel K_a is simply the linear functional of evaluation at $a \in \mathbb{U}$.

We will write $\langle f, \lambda\rangle$ for $\lambda(f)$, where $\lambda \in (H^p)^*$ and $f \in H^p$, so in particular, $\langle f, K_a\rangle = f(a)$, i.e. formula (7) still holds in this more general context. Consequently the proof of Lemma 3.2 works with almost no change to provide:

$$A^*_{\varphi,p}(K_a) \;=\; Q_\varphi(a)^{1/p}\, K_{\varphi(a)} \qquad (a \in \mathbb{U}), \tag{22}$$

where, as always for Hardy spaces, $Q_\varphi = \tau'/(\tau' \circ \varphi)$. Here the fact that $Q_\varphi(a)$ shows up unconjugated on the right-hand side of (22) reflects the fact that now our pairing "$\langle\ ,\ \rangle$" of H^p with its dual is truly bilinear, while in the Hilbert-space setting of §3 it was conjugate-linear in the second variable.

If Φ, hence φ, has an interior fixed point, then the argument employed to prove Proposition 3.3, which requires only (22) and the fact that non-zero spectral points of compact operators are eigenvalues, works almost *verbatim* in the Riesz situation. Thus the problem reduces to that of adapting the argument of §3.6 to show that if Φ has no fixed point in G, then C_Φ is not Riesz on $\mathcal{H}^p(G)$ for $1 \le p < \infty$.

To this end, suppose Φ, hence φ, has no interior fixed point, so that φ has a Denjoy-Wolff point, which we may suppose to be 1, on $\partial\mathbb{U}$. Our goal is to show that $r_e(A_{\varphi,p}) > 0$.

We require the following well-known estimate (see [5, §4.6], for example):

5.2. **Lemma.** *For each $\alpha > 1$ there exists a positive, finite constant C_α such that*

$$\int_{\partial\mathbb{U}} \frac{1}{|1 - r\zeta|^\alpha}\, dm(\zeta) \;\le\; \frac{C_\alpha}{(1-r)^{\alpha-1}} \qquad (0 \le r < 1).$$

Let $\|\cdot\|_{*,p}$ denote the norm in $(H^p)^*$. The following lemma, which is also well known, tells us how to normalize K_a in $(H^p)^*$. We present its proof in order to keep our exposition reasonably self contained.

5.3. **Lemma.** *Suppose $1 \le p < \infty$. Then there exist finite positive constants $c_1(p)$ and $c_2(p)$ such that for every $a \in \mathbb{U}$:*

$$\frac{c_1(p)}{(1-|a|^2)^{1/p}} \;\le\; \|K_a\|_{*,p} \;\le\; \frac{c_2(p)}{(1-|a|^2)^{1/p}}\ .$$

Proof. Suppose $f \in H^p$ and $a \in \mathbb{U}$. Since $f(a)$ on $\mathbb{U}$ can be computed by a Cauchy integral over $\partial\mathbb{U}$ (see [5, Theorem 3.6, page 40], for example), if $p > 1$ then Hölder's inequality yields

$$|f(a)| \;\le\; \|f\|_p \left\{ \int_{\partial\mathbb{U}} \frac{1}{|1-\overline{a}\zeta|^q}\, dm(\zeta) \right\}^{1/q},$$

where q is the index conjugate to p. By Lemma 5.2 the integral on the right is bounded by $C_q/(1-|a|)^{q-1}$, so $|f(a)| \le 2^{1/p} C_q^{1/q} \|f\|_p (1-|a|^2)^{-1/p}$, which provides the upper bound promised by the Lemma, with $c_2(p) = 2^{1/p} C_q^{1/q}$. If $p = 1$ then this bound is trivially obtained from the Cauchy integral formula, with $c_2(1) = 2$ (we remark that for $p = 2$ this bound has also been noted in (15) with $c_2(2) = 1$).

For the lower bound, let

$$f_a(z) = \left\{ \frac{(1-|a|^2)}{(1-\overline{a}z)^2} \right\}^{1/p} \qquad\qquad (z \in \mathbb{U}).$$

Then Lemma 5.2 shows that $\|f_a\|_p \le (2C_2)^{1/p}$, so

$$\frac{1}{(1-|a|^2)^{1/p}} = f_a(a) = \langle f_a,\, K_a \rangle \le \|f_a\|_p\, \|K_a\|_{*,p} \le (2C_2)^{1/p} \|K_a\|_{*,p}\,,$$

which yields the promised lower bound, with $c_1(p) = (2C_2)^{-1/p}$. □

With Lemma 5.3 in hand, we define $k_a = K_a/\|K_a\|_{*,p}$, and repeat the argument of Section 3.6—using (22) in place of Lemma 3.2, and Lemma 5.3 in place of (13)—to obtain

$$\liminf_{r\to 1-} \|A^*_{\varphi,p} k_r\|_{*,p} \ge c_3(p) \varphi'(1)^{2/p}, \tag{23}$$

where $c_3(p) = c_1(p)/c_2(p)$ (note that for the case $p = 2$ this recovers the essential content of (18)). To conclude the proof we need to know that

$$\lim_{|a|\to 1-} \|J^* k_a\|_{*,p} = 0 \qquad (\text{all } J \text{ compact on } H^p). \tag{24}$$

Granting this, we see from (23) and the reverse triangle inequality that for every such J,

$$\|A_{\varphi,p} + J\| = \|A^*_{\varphi,p} + J^*\| \ge \liminf_{r\to 1-} \|(A^*_{\varphi,p} + J^*) k_r\|_{*,p} \ge c_3(p) \varphi'(1)^{2/p}, \tag{25}$$

so that

$$\|A_{\varphi,p}\|_e \ge c_3(p) \varphi'(1)^{2/p}. \tag{26}$$

Now (26) holds, with the same constant $c_3(p)$, for *any* φ with Denjoy-Wolff point 1, hence it works as well with φ replaced by the iterate φ_n for any positive integer n. Upon noting that $A_{\varphi_n,p} = (A_{\varphi,p})^n$ and $(\varphi_n)'(1) = \varphi'(1)^n$ (by an easily proved "chain rule" for angular derivatives, see [22, §4.8, page 74, Problem 10]), we obtain

$$\|(A_{\varphi,p})^n\|_e = \|A_{\varphi_n,p}\|_e \ge c_3(p)\, \varphi'(1)^{2n/p} \qquad (n = 1, 2, \ldots),$$

from which the (essential) spectral radius formula yields

$$r_e(A_\varphi) \geq \varphi'(1)^{2/p} > 0,$$

which shows that A_φ is not a Riesz operator.

It remains to prove (24). Just as in §3.6, the upper estimate of Lemma 5.3, which is really a "big-oh" estimate on the growth of H^p functions, has a "little-oh" version which can be interpreted as saying that $k_a \to 0$ in the weak-star topology of $(H^p)^*$ as $|a| \to 1-$. If J is compact on H^p then J^* is compact on $(H^p)^*$, and so takes the collection $\{k_a : |a| < 1\}$ of unit vectors in $(H^p)^*$ into a relatively (norm-) compact subset. Now fix a sequence $\{a_n\}$ of points in $\mathbb{U}$ that converges to $\partial\mathbb{U}$. By the just-mentioned compactness there is a subsequence $\{b_j\}$ for which $\{J^* b_j\}$ converges in the norm topology of $(H^p)^*$, hence also in the weak-star topology, to some linear functional $\lambda \in (H^p)^*$. But *adjoint operators are weak-star continuous*, so it follows that as $j \to \infty$ the weak-star limit of $\{J^* k_{b_j}\}$ is zero (since $k_{b_j} \to 0$ weak star as $k \to \infty$), hence $\lambda = 0$, and so $\|J^* k_{b_j}\|_{*,p} \to 0$ as $j \to \infty$. We have shown that every sequence $\{J^* k_{a_n}\}$, with $|a_n| \to 1-$, has a subsequence that converges in the norm topology of $(H^p)^*$ to zero. From this follows (24), which completes the proof of Theorem 5.1. □

5.4. **The case $0 < p < 1$.** We mentioned in §1 that if $0 < p < 1$ then the functional $\|\cdot\|_p^p$ is a *p-norm* on $\mathcal{H}^p(G)$ (i.e. it is subadditive and homogeneous of order p), and that the metric induced on $\mathcal{H}^p(G)$ by this p-norm is complete.

Now a linear operator on such a "p-Banach space" X is continuous if and only if it is bounded on the unit ball (same proof as for ordinary Banach spaces), hence the algebra $\mathcal{L}(X)$ of all such operators can be given a p-norm in the usual way, and routine arguments show that in this "norm" it becomes a p-Banach algebra—we leave it to the reader to formulate precisely what this means. In the interest of brevity, for the rest of this section we expand the meaning of the term "norm" to include the case of p-norms.

For a p-Banach space X the compact operators $\mathcal{K}(X)$ (those which take the unit ball to a relatively compact set) still form a closed, two-sided ideal in $\mathcal{L}(X)$, and the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ survives in this setting as a p-Banach algebra. As before, the essential norm of an operator $T \in \mathcal{L}(X)$ is its distance, in the norm of $\mathcal{L}(X)$, to the compacts. We have the usual notion of "spectrum" for operators on p-Banach spaces, and more generally for elements of p-Banach algebras; in particular the essential spectrum of $T \in \mathcal{L}(X)$ is the spectrum of its coset in the Calkin algebra. The Riesz theory of compact operators goes through unchanged in this more general setting ([25], see also [17, Chapter IV, §5], [10, §21, Problem B, page 207], [23]), and there is a spectral radius formula which, when applied to the Calkin algebra, yields:

$$r_e(T) = \lim_{n\to\infty} \|T^n\|_e^{1/np} \tag{27}$$

for any $T \in \mathcal{L}(X)$ [9].

Just as before, we say that $T \in \mathcal{L}(X)$ is a *Riesz operator* if its essential spectrum is $\{0\}$, i.e., if $\lim_n \|T^n\|_e^{1/n} = 0$. The spectral properties of Riesz operators carry over from Banach spaces to this more general setting; in particular, because the Fredholm index is still continuous, non-zero spectral points are eigenvalues (see [23], for example).

Finally, we require some special facts about H^p for $0 < p < 1$. Not all p-Banach spaces are richly endowed with continuous linear functionals (for example, $L^p([0,1])$ has none but the zero functional), but H^p has enough to separate points. For example the evaluation functionals K_a $(a \in \mathbb{U})$ separate points, and all are continuous on H^p, even for $p < 1$. Let us define the norm $\|\lambda\|_*$ of a continuous linear functional λ on H^p to be the supremum of $|\lambda(f)|$ as f ranges over the unit ball of H^p. This is a *bona fide* norm on the dual space $(H^p)^*$, which makes it into a Banach space. For any linear operator continuous on H^p, the adjoint, defined in the usual way on $(H^p)^*$, is a bounded operator. While norm of this adjoint operator need no longer be equal to the norm of the original one, it follows immediately from definitions that *the norm of the adjoint operator is less than or equal to that of the original.* This is all we need to make our previous arguments work in the new setting!

Indeed, Lemma 5.3 holds for $0 < p < 1$, with the same proof. The only difference is that the growth condition $|f(a)| \leq \text{const.}\, \|f\|_p(1-|z|)^{1/p}$ can no longer be derived from a Cauchy integral representation. Nevertheless it is true, with const. $= 2^{1/p}$; see [5, Lemma, page 36] for a proof based on the factorization techniques featured in §2.4 and §2.7. With this in hand, the proof we just gave for the case $p \geq 1$ works perfectly for $0 < p < 1$; we need only replace the equality that leads off (25) by the inequality $\|A_{\varphi,p} + J\| \geq \|A^*_{\varphi,p} + J^*\|$, and check that our statements about weak-star convergence remain true. Here weak-star convergence still means: "pointwise convergence on the predual." The fact that the normalized evaluation functionals k_a converge to zero weak star as $|a| \to 1-$ follows, as before, from the "little-oh" version of the growth estimate mentioned in the second sentence of this paragraph. This result, in turn, follows as before from the original "big-oh" estimate and the density of polynomials in H^p [5, Theorem 3.3, page 36]. All that we need to complete the argument, then, is the weak-star continuity of adjoints; this follows directly from definitions. To summarize, we have proved:

5.5. **Theorem.** *Suppose that G is a simply connected domain properly contained in $\mathbb{C}$, and that $0 < p < \infty$. Then $\mathcal{H}^p(G)$ supports a Riesz composition operator if and only if ∂G has finite one-dimensional Hausdorff measure.*

6. Boundedness

Having studied the existence of compact and Riesz composition operators on the spaces $\mathcal{H}^p(G)$, we conclude with some observations about the more fundamental notion of *boundedness.* The existence of bounded composition operators is, of course, never in question, since on $\mathcal{H}^p(G)$ there is always the identity operator, which is the composition operator induced by the identity map of G. Here the question of interest is: "For which simply connected domains $G \neq \mathbb{C}$ is *every* composition operator bounded?" We noted in §1.4 that, thanks to Proposition 1.2, this happens whenever both τ' and its reciprocal are bounded on $\mathbb{U}$. In this section we show that this boundedness condition *characterizes* those domains G for which every composition operator on $\mathcal{H}^p(G)$ is bounded.

Fundamental to our proof will be the class of maps induced on G by rotations of $\mathbb{U}$. For $\omega \in \partial\mathbb{U}$ define $\Phi_\omega : G \to G$ by: $\Phi_\omega(z) = \tau(\omega\tau^{-1}(z))$ for $z \in G$. Thus Φ_ω is the holomorphic self-map of G that corresponds to rotation of $\mathbb{U}$ through the angle $\arg\omega$.

6.1. **Theorem.** *For G a simply connected domain properly contained in $\mathbb{C}$, and $0 < p < \infty$, the following four statements are equivalent:*

(a) *The operator C_{Φ_ω} is bounded on $\mathcal{H}^p(G)$ for all ω in a subset of $\partial\mathbb{U}$ having positive measure.*
(b) *C_{Φ_ω} is bounded on $\mathcal{H}^p(G)$ for all ω, and $\sup_{\omega\in\partial\mathbb{U}} \|C_{\Phi_\omega}\| < \infty$.*
(c) *The holomorphic functions τ' and $1/\tau'$ are both bounded on $\mathbb{U}$.*
(d) *The operator C_Φ is bounded on $\mathcal{H}^p(G)$ for every holomorphic selfmap Φ of G.*

Before beginning the proof we require a lemma, whose statement will surprise nobody. For convenience we switch notation and write A_ω for $A_{\varphi_\omega,p}$; hence

$$A_\omega f(z) = \left(\frac{\tau'(z)}{\tau'(\omega z)}\right)^{1/p} f(\omega z) \qquad (f \in H^p, z \in \mathbb{U}).$$

6.2. **Lemma.** *The (possibly infinite-valued) map $\omega \to \|A_\omega\|$ is measurable on $\partial\mathbb{U}$.*

Proof. For $0 \le r < 1$ and $\omega \in \partial\mathbb{U}$ define the operator $T_{\omega,r}$ on H^p by

$$T_{\omega,r} f(z) = (A_\omega f)(rz) = \left(\frac{\tau'(rz)}{\tau'(\omega rz)}\right)^{1/p} f(\omega rz) \qquad (f \in H^p, z \in \mathbb{U}).$$

Then for each $0 \le r < 1$ and $f \in H^p$ the map $\omega \to \|T_{\omega,r} f\|$ is continuous on $\partial\mathbb{U}$, hence the function

$$\omega \to \|T_{\omega,r}\| = \sup_{\|f\|\le 1} \|T_{\omega,r} f\|$$

is lower semicontinuous, and therefore measurable. Now for each $f \in H^p$, $\|T_{\omega,r} f\| \uparrow \|A_\omega f\|$ as $r \uparrow 1$, hence $\|T_{\omega,r}\| \uparrow \|A_\omega\|$ as $r \uparrow 1$ (in each case, the limit on the right may be infinite). This establishes the measurability of $\omega \to \|A_\omega\|$. □

Proof of Theorem 6.1. To keep the proof as concrete as possible, we consider only the case $p = 2$, and remind the reader that, according to Proposition 2.4, this entails no loss of generality.

(a) $\to$ (b): Let $E(G)$ be the set of points $\omega \in \partial\mathbb{U}$ such that C_{Φ_ω} is bounded on $\mathcal{H}^2(G)$, i.e., such that A_ω is bounded on H^2. Since $A_{\omega_1\omega_2} = A_{\omega_1}A_{\omega_2}$ for each pair of points $\omega_1, \omega_2 \in \partial\mathbb{U}$ we see that $E(G)$ is a subgroup of the unit circle which, by Lemma 6.2, is measurable. This measurability supplies a subset $F \subset E(G)$ of positive measure such that $\sup_{\omega\in F} \|A_\omega\| = M < \infty$. The algebraic product $F \cdot F$ contains a nontrivial arc I of the unit circle [18, Ch. 7, Problem 5, p. 156]. Each $\omega \in I$ has the form $\omega_1\omega_2$ for some $\omega_1, \omega_2 \in F$, from which it follows that

$$\|A_\omega\| = \|A_{\omega_1}A_{\omega_2}\| \le \|A_{\omega_1}\|\,\|A_{\omega_2}\| \le M^2.$$

Thus $\|A_\omega\| \le M^2$ for each $\omega \in I$. For some n the n-fold algebraic product of this arc with itself covers the whole circle, so in similar fashion, $\|A_\omega\| \le M^{2n}$ for each $\omega \in \partial\mathbb{U}$.

(b) $\to$ (c): We are assuming that $\sup_{\omega\in\partial\mathbb{U}} \|A_\omega\| = M < \infty$. Upon applying A_ω to the normalized kernel functions $k_a = K_a/\|K_a\|$ $(a \in \mathbb{U})$, where now K_a is defined by (6), the fundamental adjoint identity of Lemma 3.2 provides for each $a \in \mathbb{U}$ and $\omega \in \partial\mathbb{U}$:

$$M^2 \ge \|A_\omega^* k_a\|^2 = \frac{|\tau'(a)|}{|\tau'(\omega a)|} \ . \tag{28}$$

Given $a \in \mathbb{U}$ the maximum principle provides $\omega \in \partial\mathbb{U}$ such that $|\tau'(\overline{\omega}a)| \geq |\tau'(0)|$, hence upon using (28) with a replaced by $\overline{\omega}a$ we see that

$$\frac{1}{|\tau'(a)|} \;\leq\; M^2 \frac{1}{|\tau'(\overline{\omega}a)|} \;\leq\; \frac{M^2}{|\tau'(0)|},$$

which shows (because $\tau'(0) \neq 0$) that $1/\tau'$ is bounded on $\mathbb{U}$.

To show that τ' is bounded we fix $a \in \mathbb{U}$ and apply the maximum principle to $1/\tau'$ (holomorphic on $\mathbb{U}$ because τ' never vanishes there). This produces $\omega \in \partial\mathbb{U}$ such that $|\tau'(\omega a)| \leq |\tau'(0)|$ which, along with (28), shows that

$$|\tau'(a)| \leq M^2|\tau'(\omega a)| \leq M^2|\tau'(0)|,$$

thus establishing the boundedness of τ' on $\mathbb{U}$.

(c) $\rightarrow$ (d): As noted in §1.4, if both τ' and its reciprocal are bounded on $\mathbb{U}$ then $Q_\varphi = \tau'/(\tau' \circ \varphi)$ is bounded there also, hence A_φ is the product of the (bounded) composition operator C_φ and a bounded multiplication operator. Thus A_φ is bounded on H^2 for each φ, and therefore C_Φ is bounded on $\mathcal{H}^2(G)$ for each Φ.

(d) $\rightarrow$ (a): This implication is trivial. □

6.3. **Remarks.** (a) *Which subgroups of $\partial\mathbb{U}$ can be realized as $E(G)$?* We know that $E(G)$ is measurable, and that it can be the entire unit circle. We claim that: *Every finite subgroup of the circle is an $E(G)$ for some G.*

To see this, suppose first that $n \geq 3$ and Γ_n is a subgroup of $\partial\mathbb{U}$ of order n. It is easy to see that Γ_n must be the subgroup consisting of the n-th roots of unity. Let G be the polygon whose vertices are the elements of Γ_n, and arrange the Riemann map τ of $\mathbb{U}$ onto G so that $\tau(\omega z) = \omega\tau(z)$ for $\omega = e^{i2\pi/n}$ (and so also for every $\omega \in \Gamma_n$). Then $\tau'(z)/\tau'(\gamma z)$ is bounded on $\mathbb{U}$ if and only if $\gamma \in \Gamma_n$; this, along with (28), shows that $E(G_n) = \Gamma_n$. For $n = 2$, repeat the argument with G the "lens-shaped"domain in the unit disc that lies between a circular arc C through ± 1 of radius larger than 1, and the reflection of C in the real axis.

However we have no results beyond this; in particular: *Can an infinite subgroup of $\partial\mathbb{U}$ be $E(G)$ for some G?*

(b) *Bergman spaces.* We note in closing that the proof of Theorem 6.1 works as well for Bergman spaces, and even for weighted Bergman spaces as long as the weight w induced on $\mathbb{U}$ is radially symmetric.

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