UNUSUAL TOPOLOGICAL PROPERTIES OF THE NEVANLINNA CLASS.

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Introduction. The Nevanlinna Class N is the algebra of functions f analytic in the open unit disc whose characteristic function

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{it})|\,dt$$

is bounded for $0\leqslant r<1$. By analogy with the classical H^p spaces, a natural metric is defined on N by setting $d(f,g)=\|f-g\|$, where

$$\|f\|=\lim_{r\to 1-}\frac{1}{2\pi}\int_0^{2\pi}\log(1+|f(re^{it})|)\,dt.$$

Just as in H^p theory, d is a complete, translation invariant metric on N which induces a topology stronger than that of uniform convergence on compact subsets of the disc. But the analogy goes no further: we show here that the metric space $\mathrm{N}=(\mathrm{N},d)$ has some surprising topological properties which set it completely apart from the H^p spaces. In the first place, although N is clearly a topological group under addition, its scalar multiplication is discontinuous in the scalar variable, as noted in [1, p. 146], so it is *not* a topological vector space. We show that N is not even connected: we will see, for example, that the function

$$f(z)=\exp\frac{1+z}{1-z}$$

lies outside the component of the origin. In addition we show that the Nevanlinna Class has many linear subspaces whose relative topology is discrete. These results are best stated in the following quotient space setting. Recall that

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every f in N has a radial limit

$$f(e^{it}) = \lim f(re^{it}) \qquad (r \to 1-)$$

for almost every t; and that $\log|f(e^{it})|$ is integrable over the unit circle unless $f \equiv 0$ [4; Theorem 2.2, p. 17]. The *Smirnov Class* N^+ is the collection of functions f in N for which

$$\lim_{r \to 1-} \int_0^{2\pi} \log^+ |f(re^{it})| = \int_0^{2\pi} \log^+ |f(e^{it})|\, dt$$

(see [4; Sec. 2.5] or [10; Ch. II, Sec. 6.5, p. 82]). The class N^+ is a closed subalgebra of N, and is a topological vector space in the relative topology. Our results show that N^+ is the *largest* subspace of N which is a topological vector space in the relative topology. Since we are focusing on properties of N which are not of a linear topological nature, we divide out the linear topological properties by considering the quotient N/N^+, which is a complete, metrizable additive topological group in the quotient topology. We show that every finite dimensional linear subspace of N/N^+ is discrete, but that N/N^+ itself is not. In addition, N/N^+ has infinite dimensional discrete linear subspaces; and, like N, it is disconnected. These results pull back to N, and show that every finite dimensional subspace of N which intersects N^+ only at the origin must be discrete, while N also has infinite dimensional discrete subspaces.

Our results carry over to the Nevanlinna Classes of a large family of plane domains, including all finitely connected ones which are not just punctured planes. The main result here is that whenever a domain G is not too badly behaved near one of its boundary points, then the Nevanlinna Class of G, as defined in Rudin [11, p. 46], is disconnected.

The paper is organized into five sections, the first of which consists of background material. The next two sections treat disconnectedness and discreteness respectively, while the fourth deals with the Nevanlinna Classes of arbitrary domains. In the final section we record some open problems arising from our work; and remark on extensions of our results to functions of several complex variables, and to spaces of entire functions.

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1. Background and Notation. We denote the unit disc $|z|<1$ by U, the unit circle $|z|=1$ by T. If f is analytic in U, then $\log(1+|f|)$ is subharmonic, so the integrals

$$L(r,f)=\int_0^{2\pi}\log(1+|f(re^{it})|)\,dt$$

increase with r. Thus the (possibly infinite) limit

$$\|f\|=\lim L(f,r) \qquad (r\to 1-)$$

exists, and the inequalities

$$\log^+ x \leqslant \log(1+x) \leqslant \log 2+\log^+ x \qquad (x \geqslant 0) \tag{1.1}$$

show that f belongs to N iff $\|f\|<\infty$. The arithmetic properties of the logarithm yield inequalities

$$\begin{cases} \|f+g\| \leqslant \|f\|+\|g\| \\ \quad \|fg\| \leqslant \|f\|+\|g\| \end{cases} \tag{1.2}$$

which guarantee that N is an algebra over the complex numbers, and that the equation

$$d(f,g)=\|f-g\|$$

defines a translation invariant metric on N. From now on we use the symbol N to denote the metric space (N,d). Clearly N is a topological group under addition.

Proposition 1.1. *N is a complete metric space whose topology is stronger than that of uniform convergence on compact subsets of U.*

Proof. The comparison of topologies follows immediately from the inequality

$$\log(1+|f(z)|) \leqslant 2\|f\|/(1-|z|) \qquad (|z|\leqslant 1, f\in \mathrm{N}), \tag{1.3}$$

which in turn follows from the subharmonicity of $\log(1+|f|)$ (cf. [10, Ch. II, Sec. 3.1, p. 57]). In particular, if (f_n) is a Cauchy sequence in N, then it converges uniformly on compact subsets of U to an analytic function f, which is easily seen to lie in N. From the definition of $\|\cdot\|$ it follows readily that

$$\|f_n-f\| \leqslant \limsup \|f_n-f_m\| \qquad (m\to\infty),$$

which implies that $f_n\to f$ in N; hence N is complete.

A useful tool for studying functions of the Nevanlinna Class is the Canonical Factorization Theorem. If k is a non-negative integer, and (z_n) is a sequence in $U-\{0\}$ which satisfies the *Blaschke condition*

$$\sum_n (1-|z_n|)<\infty, \tag{1.4}$$

then the *Blaschke product*

$$B(z)=z^k\prod_n \frac{z_n-z}{1-\bar{z}_n z}\frac{|z_n|}{z_n}$$

converges uniformly on compact subsets of U to an analytic function B which has modulus <1 in U and radial limits of modulus 1 at almost every point of T [4, Sec. 2.2, pp. 18–20].

In addition to a zero of order k at the origin, B has zeroes exactly at the points (z_n) (counted according to multiplicity). In contrast, if h is a non-negative, measurable function on T whose logarithm is integrable, and ω is a complex number of modulus 1, then the *outer function*

$$F(z)=\omega\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log h(e^{it})\,dt\right\}$$

is analytic in U, non-vanishing, and has a radial limit of modulus $h(e^{it})$ for almost every t [4, Sec. 2.4, p. 24]. In fact the definition shows that the reciprocal of an outer function is again outer. Intermediate between outer functions and Blaschke products are the *singular inner functions*, defined by

$$S_\mu(z)=\exp\left\{\int\frac{z+e^{it}}{z-e^{it}}\,d\mu(t)\right\} \tag{1.5}$$

where μ is a finite, positive, singular Borel measure on T, henceforth referred to as the *measure associated with* S_μ. Clearly S_μ is analytic, non-vanishing, and bounded by 1 in U. Moreover S_μ has radial limits of modulus 1 at almost every point of T. Thus singular inner functions have properties in common with both Blaschke products and outer functions. In our work it will be *reciprocals* of singular inner functions which play the dominant role.

It is easy to see that Blaschke products, outer functions, and quotients of singular inner functions all belong to the Nevanlinna Class. In fact, each f in N is representable as a product of such functions.

Canonical Factorization Theorem [4, Thm. 2.9, p. 25], [10, Ch. II, Sec. 6, p. 79]. If $f\in N$ then there exists a unique Blaschke product B, outer function F, and unique singular inner functions S_μ and S_ν with mutually singular associated measures such that $f=B(S_\nu/S_\mu)F$.

We will frequently use the notation $\mu = \mu_f$ and $\nu = \nu_f$ to identify the measures which occur respectively in the denominator and numerator of the canonical factorization of the function f.

We close this section with some remarks about the space N^+. Recall that f belongs to N^+ iff $f \in N$ and

$$\lim_{r \to 1-} \int_0^{2\pi} \log^+ |f(re^{it})|\, dt = \int_0^{2\pi} \log^+ |f(e^{it})|\, dt.$$

We will need the following alternate characterization of N^+ (cf. [4, Sec. 2.5], [10, Ch. V, Lemma 2.1, p. 123]).

PROPOSITION 1.2. *For $f \in N$ the following are equivalent*:
(a) *f belongs to* N^+,
(b) $\mu_f(T) = 0$,
(c) $\lim_{r \to 1-} \int_0^{2\pi} \log(1 + |f(re^{it})|) = \int_0^{2\pi} \log(1 + |f(e^{it})|)\, dt.$

Proof. The equivalence of (a) and (b) is in [4, Sec. 2.5], and we omit it. From [4, Ch. 2, p. 30, Problem 9] we see that $f \in N^+$ iff $\log^+|f(re^{it})| \to \log^+|f(e^{it})|$ in $L^1(T)$ as $r \to 1-$. This fact, inequalities (1.1), and the Vitali Convergence Theorem [12, Ch. 6, p. 134, Problem 9] establish the equivalence of (a) and (c).

COROLLARY. N^+ *is a closed linear subspace of* N, *and a topological vector space in the relative topology.*

Proof. That N^+ is a linear subspace of N (in fact a sub-algebra) follows readily from Proposition 1.2 ($a \leftrightarrow b$) and the Canonical Factorization Theorem. The equivalence of (a) and (c) shows that for each f in N^+ we have

$$\|f\| = \frac{1}{2\pi} \int_0^{2\pi} \log(1 + |f(e^{it})|)\, dt;$$

so the dominated convergence theorem shows that on N^+ scalar multiplication is continuous in the scalar variable. Routine arguments like those employed in L^p spaces show that N^+ is complete, so it is an F-space in the sense of Banach, and therefore a topological vector space [2, II.1.12, p. 53].

We remark that the metric d induces on N^+ the same topology as the one defined in Gamelin [6; Chapter V, p. 122], so the preceding corollary is just a special case of [6, Chapter V, Theorem 2.3, p. 123]. The linear topological structure of N^+ has recently been studied in some detail by N. Yanagihara [15], [16], while C. S. Davis [1] has studied iterated radial limits for the analogous class defined on a polydisc in $\mathbf{C}^n$.

2. Disconnectedness. In this section we show that both N and its quotient N/N^+ are disconnected. In particular each function

$$f_a(z) = \exp a\frac{1+z}{1-z} \qquad (a>0) \tag{2.1}$$

lies outside the component of the origin, and more generally so does every f in N whose denominator measure has a mass point. The idea of the proof is to find a non-trivial subadditive, continuous, "non-archimedian" functional on N.

More precisely, for ω on the unit circle we define $\lambda = \lambda_\omega$ on N by

$$\lambda_\omega(f) = \lambda(f) = \limsup_{r\to 1-} (1-r)\log^+|f(r\omega)|. \tag{2.2}$$

The elementary inequality

$$\log^+(x+y) \leqslant \log^+ x + \log^+ y + \log 2 \qquad (x,y \geqslant 0)$$

shows that λ is subadditive; while inequality (1.3) yields

$$\lambda(f) \leqslant 2\|f\| \qquad (f \text{ in N}) \tag{2.3}$$

from which it follows that λ is continuous. It is easy to see from (2.2) that

$$\lambda(f+g) = \max\{\lambda(f),\lambda(g)\} \qquad \text{if } \lambda(f)\neq\lambda(g). \tag{2.4}$$

Henceforth we refer to this as the *non-archimedian property* of λ.

THEOREM 2.1. N *is disconnected.*

Proof. We actually show that if $\lambda(f)\neq\lambda(g)$ then f and g lie in different components of N. In particular it is easy to see that if f_a is defined by (2.1) and $\omega = 1$, then $\lambda(f_a) = 2a$; hence the functions f_a $(a>0)$ all lie in different components of N, and none of them belong to the component of the origin.

So suppose $f, g \in N$ and $\lambda(g) < a < \lambda(f)$. Then the set

$$V = \{h \in N : \lambda(h) > a\}$$

contains f but not g and is open because λ is continuous. We need only show that V is closed. Fix $h_0 \in N$, $\notin V$. Then for each h in V we have

$$\lambda(h_0) \leqslant a < \lambda(h) = \lambda(-h),$$

so (2.3) and the non-archimedian property (2.4) yield

$$2\|h_0 - h\| \geqslant \lambda(h_0 - h) > a.$$

In other words, the open ball of radius $a/2$ about h_0 lies entirely outside of V. Thus V is closed, and the proof is complete.

COROLLARY 1. N *is not a topological vector space.*

Now N^+ is a topological vector space, so it lies in the component of N which contains the origin. In fact we conjecture that N^+ *is* the component of the origin, but have been unable to prove this. The proof of Theorem 2.1 established that if $\lambda_\omega(f)\neq 0$ for some $\omega\in T$, then f lies outside the component of the origin. This immediately yields:

COROLLARY 2. $\lambda_\omega(f)=0$ *for each* $f\in N^+$ *and* $\omega\in T$.

Later in this section we will see that $\lambda_\omega(f)=2\mu_f\{\omega\}$, thus providing another proof of this corollary. Right now, however, we want to use it to show that N/N^+ is disconnected.

For $f\in N$ let $\bar{f}$ denote the coset of $f+N^+$ in N/N^+, and let

$$\|\bar{f}\|=\inf\{\|g\|:g\in\bar{f}\}.$$

Then the translation invariant metric

$$\bar{d}(\bar{f},\bar{g})=\|\bar{f}-\bar{g}\|$$

induces the quotient topology on N/N^+. It is well known that $N/N^+=(N/N^+,\bar{d})$ is a complete additive topological group [14, Theorem 12.3.5, p. 264]. Since the quotient map $f\to\bar{f}$ is continuous, the next result actually implies Theorem 2.1.

THEOREM 2.1′. N/N^+ *is disconnected.*

Proof. Fix $\omega\in T$ and let $\lambda=\lambda_\omega$ as in (2.2). Since λ is subadditive and vanishes on N^+ (Corollary 2), it is constant on each coset $\bar{f}$, so the equation

$$\Lambda(\bar{f})=\lambda(f)\qquad(f\in N)$$

makes sense, and defines a non-trivial subadditive functional on N/N^+ which has the non-archimedian property. By estimate (2.3) we have

$$\Lambda(\bar{f})\leq 2\|\bar{f}\|\qquad(f\in N),$$

so Λ is continuous. The disconnectedness of N/N^+ now follows just as for N, and the proof is complete.

Of course this proof shows that if f_a is given by (2.1), then the cosets $\bar{f}_a$

$(a>0)$ all lie in different components of N/N^+, and none lies in the component of the origin. Our conjecture about the component of the origin in N projects onto one about N/N^+.

CONJECTURE. N/N^+ *is totally disconnected.*

Our next goal is to connect the functionals λ_ω with the Canonical Factorization Theorem. Recall from section 1 that if $f=B(S_\nu/S_\mu)F$ is the Canonical Factorization of $f\in N$, then the denominator measure μ is denoted by μ_f.

THEOREM 2.2. *For* $f\in N$ *and* $\omega\in T$, $\lambda_\omega(f)=2\mu_f\{\omega\}$.

In particular since λ_ω (resp. $\bar{\lambda}_\omega$) is constant on components of N (resp. N/N^+), Theorem 2.2 yields the following:

COROLLARY. *Suppose* $f,g\in N$ *and* $\mu_f\{\omega\}\neq\mu_g\{\omega\}$ *for some* $\omega\in T$. *Then* f *and* g *lie in different components of* N, *and* $\bar{f}$ *and* $\bar{g}$ *lie in different components of* N/N^+.

The proof of Theorem 2.2 requires an auxiliary estimate of the rate at which a Blaschke product can decay on a radius. This estimate, which is of independent interest, occurs in the work of M. Heins [8, Theorem 6.1, p. 193] in an apparently weaker, but actually equivalent form.

LEMMA 2.3. *If* B *is a Blaschke product, then*

$$\limsup_{r\to 1-}(1-r)\log|B(r)|=0$$

For completeness we will give an elementary proof of this result at the end of this section. But first we use it to prove Theorem 2.2.

Proof of Theorem 2.2. Without loss of generality we can take $\omega=1$. It is convenient to consider the functional

$$l(f)=\limsup_{r\to 1-}(1-r)\log|f(r)| \qquad (f\in N),$$

and the measure

$$\sigma_f=\mu_f-\nu_f$$

where μ_f and ν_f are respectively the denominator and numerator measures of f, as defined in Section 1. Since μ_f and ν_f are mutually singular, at most one of them can have any mass at 1, so $\mu_f\{1\}=\max(\sigma_f\{1\},0)$. But $\lambda(f)=\max(l(f),0)$, so to prove the theorem it is enough to show that

$$l(f)=2\sigma_f\{1\} \qquad (f\in N). \tag{2.5}$$

We are going to show that if $f \in \mathrm{N}$ has no zeroes in the open disc U then a bit more is true:

$$\lim_{r \to 1-} (1-r)\log|f(r)| = 2\sigma_f\{1\}. \tag{2.6}$$

This will yield (2.5) for any $f \in \mathrm{N}$, since if f has zeroes in U, then by the Canonical Factorization Theorem, $f = Bg$, where B is a Blaschke product, $g \in \mathrm{N}$, g has no zeroes in U, and $\sigma_f = \sigma_g$. Since

$$\log|f| = \log|B| + \log|g|,$$

equation (2.6) and Lemma 2.3 will then yield

$$l(f) = l(g) = 2\sigma_g\{1\} = 2\sigma_f\{1\},$$

as desired.

So it remains to establish equation (2.6). If $f \in \mathrm{N}$ and f never vanishes in U, then by the Canonical Factorization Theorem, the harmonic function

$$u(z) = \log|f(z)|$$

is the Poisson integral of the measure

$$d\rho(t) = d\sigma_f(t) + \log|f(e^{it})|\,dt/2\pi.$$

We have to show that

$$\lim (1-r)u(r) = 2\sigma_f\{1\} \qquad (r \to 1-). \tag{2.7}$$

This follows from a standard argument which we state here for completeness. Let $a = \sigma_f\{1\}$ and let $\rho_0 = \rho - a\delta$ where δ is the unit mass at the point 1. Letting v denote the Poisson integral of ρ_0 we have

$$u(z) = v(z) + a\,\mathrm{Re}\,\frac{1+z}{1-z} \qquad (z \text{ in } U)$$

so to prove (2.7) we need only show that

$$\lim (1-r)v(r) = 0 \qquad (r \to 1-). \tag{2.8}$$

Fix $\varepsilon > 0$ and choose an open arc I of the unit circle centered at the point 1 such that $|\rho_0(I)| < \varepsilon$: this is possible because ρ_0 has no mass at 1. Since the

Poisson Kernel $P_r(e^{it})$ tends to zero uniformly on $T-I$ we have

$$\begin{aligned} 2\pi|v(r)| &= \left|\int_T P_r d\rho_0\right| \\ &\leqslant 0(1) + \int_I P_r d|\rho_0| \\ &\leqslant 0(1) + 2\varepsilon/(1-r) \end{aligned}$$

as $r \to 1-$, which completes the proof.

Proof of Lemma 2.3. We can clearly assume that $B(0) \neq 0$. In addition we can assume that the zeroes of B lie on the positive real axis. Indeed, if this is not the case, then let (z_n) denote the zeroes of B, counted according to multiplicity, and let $B_0(z)$ be the convergent Blaschke product formed with the sequence $(|z_n|)$. An easy calculation shows that

$$\left|\frac{|z|-|z_n|}{1-|z_n z|}\right| \leqslant \left|\frac{z-z_n}{1-\bar{z}_n z}\right|,$$

hence $|B_0(r)| \leqslant |B(r)|$ for $0 \leqslant r < 1$. Thus if we can prove Lemma 2.3 for B_0, then we also have it for B (this is a special case of the "Contraction Principle" employed in [8]).

So suppose that the zeroes of B lie in the open unit interval, and that the conclusion of the lemma fails for B. Then there exists a constant $A > 0$ such that

$$|B(r)| < \exp\left\{-A\frac{1+r}{1-r}\right\} \qquad (0 \leqslant r < 1). \tag{2.9}$$

We are going to show that

$$|B(z)| < \exp\left\{-A \operatorname{Re}\frac{1+z}{1-z}\right\} \qquad (|z| < 1). \tag{2.10}$$

To see how this gives the desired result, let S be a Stolz region in U with vertex at 1, and note that (2.10) implies that there exists $a > 0$ such that

$$|B(z)| \leqslant \exp[-a/(1-|z|)] \tag{2.11}$$

for all z in S. Let S_r be the intersection of S with the circle $|z| = r$ $(0 < r < 1)$. Then (2.11) yields

$$\begin{aligned} \int_0^{2\pi} \log|B(re^{it})|\,dt &\leqslant \int_{S_r} \log|B(re^{it})|\,dt \\ &\leqslant -a|S_r|/(1-r) \end{aligned}$$

for $0<r<1$, where $|S_r|$ is the length of the arc S_r. But $|S_r| \geqslant b(1-r)$ for some constant $b>0$, so for $0<r<1$:

$$\int_0^{2\pi} \log|B(re^{it})|\,dt \leqslant -ab<0.$$

But this contradicts the fact that for every Blaschke product B,

$$\lim_{r\to 1-} \int_0^{2\pi} \log|B(re^{it})|\,dt=0$$

[10; Sec. 7.2, p. 51], so the proof will be complete once we verify (2.10).

To prove (2.10), map the unit disc onto the right half-plane by the transformation

$$w=\frac{1+z}{1-z},$$

and let $C(w)=B(z)$. Then $|C| \leqslant 1$ in the right half-plane, and (2.9) shows that

$$|C(x)| \leqslant e^{-Ax}$$

for all positive x. If $\operatorname{Re} w \geqslant 0$, let

$$f(w)=e^{Aw}C(w).$$

Since the zeroes of the Blaschke product B lie in the open unit interval, the product converges uniformly in a neighborhood of each point of $T-\{1\}$. Thus C is actually continuous in the closed right half-plane (in fact it is analytic across the entire imaginary axis), hence so is f. Moreover, $|f|$ is identically 1 on the imaginary axis, $\leqslant 1$ on the positive real axis, and $\leqslant e^{A|w|}$ for all w in the right half-plane. Thus the Phragmén-Lindelöf Theorem [12; Ch. 12, p. 250, Problem 6] applies to f in the first and fourth quadrants respectively, and shows that $|f| \leqslant 1$ in both, hence in the entire right half-plane. Thus

$$|C(w)| \leqslant e^{-\operatorname{Re} w}$$

in the right half-plane, and this transforms back into the desired inequality (2.10); which completes the proof.

We remark that the argument used to move the zeroes of B onto the open unit interval could have been omitted at the expense of using a more sophisticated version of the Phragmén-Lindelöf Theorem [5; Ch. 6, Theorem 6.6, p. 77].

3. Discreteness. Suppose that f is a function in the Nevanlinna Class whose denominator measure has a mass point: $\mu_f\{1\}>0$, say. Clearly $\mu_{af}=\mu_f$ for any scalar a, so if $\lambda=\lambda_1$ is defined as in (2.2), we obtain from Theorem 2.2 and estimate (2.3) the inequality

$$0<2\mu_f\{1\}=\lambda(af)\leqslant 2\|af\|,$$

which in turn yields

$$\lim_{a\to 0}\|af\|\geqslant \mu_f\{1\}. \tag{3.1}$$

It follows from (3.1) and the translation invariance of the metric on N that the one dimensional linear subspace spanned by f has the discrete topology.

In this section we prove that N has many more discrete subspaces: for example every finite dimensional subspace which intersects N^+ only at the origin is discrete, and there are also infinite dimensional discrete subspaces. These facts follow from the next result, which improves estimate (3.1).

THEOREM 3.1. *If $f\in N$ then $\lim_{a\to 0}\|af\|=\mu_f(T)$.*

We prefer to discuss the consequences of the theorem before giving its proof. The first one follows immediately from the theorem and Proposition 1.2 $(a\leftrightarrow b)$.

COROLLARY 1. *Suppose $f\in N$. Then $f\in N^+$ iff the one dimensional linear subspace spanned by f has the discrete topology.*

With a bit more effort we can improve this to:

COROLLARY 2. *Every finite dimensional linear subspace of N/N^+ has the discrete topology.*

Proof. We define a preliminary functional m on N by

$$m(f)=\mu_f(T)=\lim_{a\to 0}\|af\|, \tag{3.2}$$

where the last inequality is, of course, Theorem 3.1. It follows that m is subadditive, $m(f)\leqslant\|f\|$, and $m(af)=m(f)$ for each f in N and non-zero scalar a. Clearly $m=0$ on N^+ (Proposition 1.2), so it follows from subadditivity that m is constant on cosets $\bar{f}=f+N^+$ (f in N). Thus the equation

$$M(\bar{f})=m(f)\qquad (f\text{ in N})$$

defines a subadditive functional on N/N^+ which vanishes only at the origin and has the additional properties

$$M(\bar{f}) \leqslant \|\bar{f}\| \tag{3.3}$$

and

$$M(a\bar{f}) = M(\bar{f}) \tag{3.4}$$

for each f in N and non-zero scalar a.

Let S be a finite dimensional linear subspace of N/N^+, and define

$$\delta(\mathrm{S}) = \inf\left\{M(\bar{f}) : \bar{f} \text{ in } \mathrm{S}\right\}. \tag{3.5}$$

We want to show that

$$\inf\left\{\|\bar{f}\| : \bar{f} \text{ in } \mathrm{S}\right\} > 0,$$

so by (3.3) it is enough to prove that $\delta(\mathrm{S}) > 0$. If $\dim \mathrm{S} = 1$ this follows immediately from Theorem 3.1 and the definition of M. We proceed by induction. Suppose that (3.5) is true for every n dimensional subspace of N/N^+. Suppose S is $n+1$ dimensional, choose an n dimensional subspace V of S, and fix s in S, $s \not\in V$. By the induction hypothesis $\delta(V) > 0$: we want to show that $\delta(\mathrm{S}) > 0$ also. Suppose not. Then there is a sequence (v_j) in V and a sequence (a_j) of scalars such that

$$\lim M(a_j\mathrm{S} + v_j) = 0 \qquad (j \to \infty). \tag{3.6}$$

By the induction hypothesis at most finitely many of the a_j can be zero, so we may assume that none of them are. Let $w_j = a_j^{-1}v_j$. Then $w_j \in V$, so by (3.4) and (3.6) we have

$$\lim M(s + w_j) = 0 \qquad (j \to \infty). \tag{3.7}$$

Now the subadditivity of M yields

$$0 \leqslant M(w_i - w_j) \leqslant M(w_i + s) + M(w_j + s),$$

so for all i and j sufficiently large, we have from (3.7) that

$$M(w_i - w_j) < \delta(V).$$

Since $w_i - w_j \in V$, this implies that for some w in V,

$$w_i = w_j = w$$

for all i,j sufficiently large. So $M(s+w)=0$ by (3.7), and since M vanishes only at the origin we must have $s=-w\in V$, contradicting our choice of s. Thus $\delta(S)>0$, and we are done.

Corollary 2 pulls back to N without difficulty, with the following result.

COROLLARY 3. *Every finite dimensional linear subspace of* N *which intersects* N^+ *only at the origin has the discrete topology.*

Proof. Suppose S is a finite dimensional subspace of N, and $\mathrm{S}\cap\mathrm{N}^+=\{0\}$. Then the quotient map $f\to\bar{f}$ is a one-to-one continuous transformation taking S onto a subspace of N/N^+ which is finite dimensional, and hence by Corollary 2, discrete. It follows easily that S itself must be discrete.

Of course Corollary 2 would hold trivially if N/N^+ itself were discrete. Fortunately this is not the case.

COROLLARY 4. N/N^+ *is not discrete.*

Proof. We need only find a non-zero sequence in N/N^+ converging to zero. Let

$$g_n(z)=\exp\frac{1}{n}\left(\frac{1+z}{1-z}\right)\qquad(n>0)$$

(so $g_n=f_{(1/n)}$ in the notation of (2.1)). By Theorem 3.1

$$\lim_{a\to0}\|ag_n\|=\frac{1}{n}$$

so we can find non-zero scalars a_n such that $\|a_ng_n\|\to0$. Now let $h_n=a_ng_n$. Then

$$0<\|\bar{h}_n\|\leqslant\|h_n\|\to0$$

as $n\to\infty$, which completes the proof.

We now present Davis' proof of Theorem 3.1. It hinges on the following standard characterization of the Nevanlinna Class which is an easy consequence of the Canonical Factorization Theorem and [13, Theorem 3.3.5, p. 46].

LEMMA 3.2. *A function f analytic in the open unit disc belongs to the Nevanlinna Class iff $\log^+|f|$ has a harmonic majorant. If $f\in N$ then the least harmonic majorant of $\log^+|f|$ coincides with the Poisson integral of the measure*

$$d\rho_f(t)=\log^+|f(e^{it})|dt/2\pi+d\mu_f, \tag{3.8}$$

where μ_f is the denominator measure of f.

We also need the fact that if w is a subharmonic function in U which has a harmonic majorant, then its least harmonic majorant W is the Poisson integral of a measure ω on T, and

$$\lim_{r\to 1-}\frac{1}{2\pi}\int_0^{2\pi} w(re^{it})\,dt = W(0) = \omega(T).$$

If in addition w has radial limits

$$w(e^{it}) = \lim_{r\to 1-} w(re^{it})$$

for almost all t, then the absolutely continuous part of $d\omega$ is $w(e^{it})dt/2\pi$ [13, Sec. 3.2].

Proof of Theorem 3.1. Fix $f\in \mathrm{N}$. We will denote the least harmonic majorant of $\log^+|f|$ (resp. $\log(1+|f|)$) by $v[f]$ (resp. $w[f]$). If μ is a finite Borel measure on T we will denote its Poisson integral evaluated at z by $P[d\mu](z)$. Thus $v[f]=P[d\rho_f]$, where ρ_f is the measure in equation (3.8), and $w[f] = P[d\beta_f]$ for some positive Borel measure β_f.

Now it follows from (1.1) that

$$v[f]\leqslant w[f]\leqslant v[f]+\log 2,$$

that is,

$$P[d\rho_f]\leqslant P[d\beta_f]\leqslant P[d\rho_f+(\log 2)d\theta/2\pi]$$

so the same inequality must hold among the respective measures, and therefore among their singular parts. This shows that ρ_f and β_f have the same singular part, which by (3.8) is μ_f, the denominator measure of f.

Now suppose that $0<a\leqslant 1$. Then a little arithmetic yields

$$\log a+\log(1+|f|)\leqslant \log(1+a|f|)\leqslant \log(1+|f|),$$

so

$$\log a+w[f]\leqslant w[af]\leqslant w[f],$$

from which it follows, as in the last paragraph, that the singular part of β_f (which is μ_f) coincides with that of β_{af}. By the remark following the statement of Lemma 3.2 we obtain from all of this:

$$d\beta_{af}(t)=\log(1+a|f(e^{it})|)dt/2\pi+d\mu_f(t),$$

hence

$$\|af\| = \beta_{af}(T) = \frac{1}{2\pi}\int_0^{2\pi} \log(1 + a|f(e^{it})|)\,dt + \mu_f(T).$$

By the Lebesgue Dominated Convergence Theorem the integral on the right tends to zero as $a \to 0+$, so

$$\lim_{a\to 0+} \|af\| = \mu_f(T)$$

which completes the proof.

We close this section by noting that the finite dimensional subspaces of N/N^+ are not the only ones with the discrete topology.

PROPOSITION 3.3. *N/N^+ has infinite dimensional discrete linear subspaces.*

Proof. For each $\omega \in T$ let

$$g_\omega(z) = \exp \frac{\omega + z}{\omega - z}.$$

Clearly the cosets $\bar{g}_\omega$ are linearly independent as elements of N/N^+, so they span an infinite dimensional linear subspace S. We claim that S has the discrete topology.

To see this, suppose that $\omega_1, \ldots, \omega_n$ are distinct points of T, and $a_1, \ldots, a_n$ are non-zero scalars. Then

$$\bar{g} = \sum_{j=1}^{n} a_j \bar{g}_{\omega_j} \neq \bar{0}.$$

Without loss of generality we can assume that $\omega_1 = 1$. Since the functions $g_{\omega_2}, \ldots, g_{\omega_n}$ are each bounded in a neighborhood of 1, we have as in the proof of Theorem 2.1′ that

$$2\|\bar{g}\| \geqslant \wedge(\bar{g}) = \limsup_{r\to 1-} (1-r)\log^+|g(r)| = 2$$

(the lim sup is actually a limit in this case). Thus $\|\bar{g}\| \geqslant 1$ for every non-zero $\bar{g}$ in S, which shows that S has the discrete topology and completes the proof.

4. Arbitrary Domains. For a plane domain G the Nevanlinna Class $N(G)$ is the algebra of functions f analytic on G for which $\log^+|f|$, or equivalently $\log(1+|f|)$ has a harmonic majorant [11; Sec. 1.5, p. 48]. Denoting the least harmonic majorant of $\log(1+|f|)$ by $V[f]$, we again exploit the analogy with H^p spaces [11, Sec. 2.6, pp. 49–50] and define

$$\|f\| = V[f](p), \tag{4.1}$$

where p is a point in G which is fixed for the remainder of the discussion. This functional induces a metric which turns $\mathrm{N}(G)$ into a complete additive topological group (see [9, p. 11] for the analogous proof for H^p spaces). Harnack's inequality shows that the topology on $\mathrm{N}(G)$ does not depend on the choice of p. The spaces $\mathrm{N}(G)$ are conformally invariant in that if φ is a one-to-one conformal map of G onto a domain G', and $p' = \varphi(p)$ is the point used in the definition of the metric on $\mathrm{N}(G')$, then the composition operator $f \to f \circ \varphi$ is an isometric isomorphism of $\mathrm{N}(G')$ onto $\mathrm{N}(G)$. If G is the unit disc and $p = 0$, then (4.1) is the same functional used to metrize $\mathrm{N} = \mathrm{N}(U)$ in the previous sections.

In this section we carry the results of Secs. 2 and 3 over to a large class of plane domains having non-trivial Nevanlinna Classes. Among these are all finitely connected domains which are not just punctured planes.

We begin by recalling that a non-negative harmonic function on G is *quasi-bounded* if it is a pointwise increasing limit of non-negative bounded harmonic functions [9, p. 7]. The quasi-bounded harmonic functions in the unit disc are precisely the Poisson integrals of non-negative integrable functions on the circle, and it is easy to see that $f \in \mathrm{N}^+$ iff $\log^+|f|$ has a quasi-bounded harmonic majorant on the disc. This motivates the following:

Definition. $\mathrm{N}^+(G)$ *is the class of functions* f *analytic in* G *such that* $\log^+|f|$ *has a quasi-bounded harmonic majorant.*

As in the case of the unit disc, $\mathrm{N}^+(G)$ is a closed subalgebra of $\mathrm{N}(G)$. In fact most of these elementary results for $\mathrm{N}(G)$ and $\mathrm{N}^+(G)$ can be deduced from the case $G = U$ by standard uniformization arguments, [11; Sec. 2.2] which we briefly present now. It is well known that $\mathrm{N}(G)$ contains non-constant functions iff the boundary of G has positive logarithmic capacity [11; Theorem 1.6, p. 48]. In this case the Uniformization Theorem [7; Chapter VI, Sec. 1] provides a covering map φ taking U onto G with $\varphi(0) = p$, and a covering group Γ consisting of all conformal automorphism of U which leave φ invariant. For $f \in \mathrm{N}(G)$ the function $\tilde{f} = f \circ \varphi$ is in $\mathrm{N}(U)$, and is Γ-invariant:

$$\tilde{f}(g(z)) = \tilde{f}(z) \qquad (g \text{ in } \Gamma, \quad z \text{ in } U),$$

and the map $f \to \tilde{f}$ is an isometric isomorphism of $\mathrm{N}(G)$ onto the closed subalgebra of $\mathrm{N}(G)$ consisting of Γ-invariant functions (see [4; Sec. 10.5] or [11; Sec. 2.2] for the analogous proofs for H^p space). The same map takes $\mathrm{N}^+(G)$

onto the subalgebra of Γ-invariant functions in N^+. This follows from the fact that a non-negative harmonic function u on G is quasi-bounded iff $\tilde{u} = u \circ \varphi$ is quasi-bounded on U. Clearly $\tilde{u}$ is quasi-bounded if u is. Conversely, if $\tilde{u}$ is quasi-bounded, then $\tilde{u} = \lim \beta_n$, where β_n is the greatest harmonic minorant of $v_n = \min(\tilde{u}, n)$. Since v_n is Γ-invariant, so is β_n (cf. [11; proof of Theorem 1.6]), so $\beta_n = \tilde{b}_n$ for some non-negative bounded harmonic function b_n on G, and $b_n \uparrow u$, which completes the proof.

Since $N(G)$ and $N^+(G)$ can be identified in this way with closed subspaces of N and N^+ respectively, their completeness is immediate, as is the fact that $N^+(G)$ is a topological vector space. In addition, Theorem 3.1. yields an immediate corollary, which we state as

PROPOSITION 4.1. *Suppose* $f \in N(G)$. *Then*

$$\lim_{a \to 0} \|af\| = 0 \qquad \text{iff } f \in N^+(G).$$

The proof of Corollaries 2 and 3 of Theorem 3.1 now yield:

COROLLARY. *Every finite dimensional linear subspace of* $N(G)/N^+(G)$ *is discrete, as is every finite dimensional subspace of* $N(G)$ *which intersects* $N^+(G)$ *only at the origin.*

The trouble with these results is that we do not know if $N(G)$ is always different from $N^+(G)$ whenever $N(G)$ is non-trivial. The next result shows that as long as some part of the boundary of G is not too pathological then $N(G) \neq N^+(G)$: in fact $N(G)$ is disconnected. More precisely, we call a boundary point ζ of G *normal* if it lies in a non-degenerate boundary component K, and is not a limit of points lying in other boundary components.

THEOREM 4.2. *If the boundary of G has a normal point then* $N(G)$ *is disconnected.*

Proof. As in the proof of Theorem 2.1, we need only construct a non-trivial continuous, subadditive functional λ on $N(G)$ with the non-archimedian property (2.4). Suppose the normal point ζ of G belongs to a boundary component K. By a preliminary conformal mapping of $\mathbf{C} - K$ onto the unit disc we may assume that $G \subset U$, K is the unit circle, and $\zeta = 1$ (recall that $N(G)$ and $N^+(G)$ are conformally invariant). So there is an open disc of radius $\varepsilon > 0$ centered at 1 whose intersection with U contains no point of the boundary of G. In particular the line segment $(1-\varepsilon, 1)$ lies entirely in G, so it makes sense to define

$$\lambda(f) = \limsup_{r \to 1-} (1-r) \log^+ |f(r)|$$

for $f \in \mathrm{N}(G)$. Letting $\rho = 1 - \varepsilon/2$ and applying Harnack's inequality in the disc $|z-\rho| < 1-\rho$ we obtain

$$u(r) < 2u(\rho)/(1-r) \qquad (\rho < r < 1)$$

for each non-negative function u harmonic in that disc. In particular if $f \in \mathrm{N}(G)$ and $v[f]$ is the least harmonic majorant of $\log^+|f|$, then

$$\log^+|f(r)| \leqslant 2v[f](\rho)/(1-r) \qquad (\rho < r < 1).$$

Now choose the point p used to define the metric in $\mathrm{N}(G)$ (Eq. 4.1) to be ρ. Then the last inequality yields

$$\lambda(f) \leqslant 2\|f\| \qquad (f \in \mathrm{N}(G)), \tag{4.2}$$

so at least λ is finite on $\mathrm{N}(G)$. It is easy to see that λ has the non-archimedian property (2.4); and that λ is subadditive, hence continuous by (4.2). So we need only find a function f in $\mathrm{N}(G)$ such that $\lambda(f) \neq 0$, and as usual we turn to

$$f(z) = \exp\frac{1+z}{1-z}.$$

Since $\lambda(f) = 2$, the proof is complete.

COROLLARY. *If G is a finitely connected domain other than a punctured plane, then $N(G)$ is disconnected.*

5. Further Remarks and Open Problems. The phenomena we have observed in the Nevanlinna Classes of plane domains can also be studied in several complex variables. For example consider the polydisc

$$U^n = U \times U \times \cdots \times U \qquad (n \text{ times})$$

and the ball

$$B^n = \{(z_1, \ldots, z_n) : |z_1|^2 + \cdots + |z_n|^2 < 1\}$$

in $\mathbf{C}^n$. The classes N and N^+ can be defined for these domains just as in one variable, with integration over the circle $|z| = r$ replaced by integration over the polycircle $|z_n| = r$ $(1 \leqslant j \leqslant n)$ or the sphere $|z_1|^2 + \cdots + |z_n|^2 = r$ respectively. If $f \in \mathrm{N}(U^n)$ then the least n-harmonic majorant of $\log^+|f|$ is the Poisson integral of a positive measure ν_f on the distinguished boundary T^n, and $f \in \mathrm{N}^+(U^n)$ iff this measure is absolutely continuous with respect to Lebesgue measure on T^n [13; Theorem 3.3.5, p. 46]. A similar statement holds for $\mathrm{N}(B^n)$ with "harmonic" replacing "n-harmonic," and the topological boundary of B^n replacing T^n, [3; Theorem 4.5, p. 125]. In both of these cases, Davis' proof of Theorem 3.1 shows that

$$\lim_{a \to 0} \| af \| = \text{total variation of } \mu_f, \tag{5.1}$$

where μ_f is the singular part of ν_f. Corollaries 1–3 of Theorem 3.1 therefore hold for $\mathrm{N}(B^n)$ and $\mathrm{N}(U^n)$. For B^n, however, we encounter the same problem we had with arbitrary plane domains: we do not know if $\mathrm{N}(B^n)$ is different from $\mathrm{N}^+(B^n)$.

Problem 5.1. (a) Is $\mathrm{N}(B^n) \neq \mathrm{N}^+(B^n)$ for $n > 1$? (b) If G is a plane domain for which $\mathrm{N}(G)$ is non-trivial, is $\mathrm{N}(G) \neq \mathrm{N}^+(G)$?

No such difficulties arise with $\mathrm{N}(U^n)$. In fact the map P defined by

$$Pf(z) = f(z, 0, 0, \ldots, 0) \qquad (f \in \mathrm{N}(U^n))$$

takes $\mathrm{N}(U^n)$ continuously onto $\mathrm{N}(U)$, so it follows from the disconnectedness of $\mathrm{N}(U)$ that $\mathrm{N}(U^n)$ is also disconnected.

Problem 5.2. What is the component of the origin in $\mathrm{N}(U^n)$?

We conjecture that the answer is $\mathrm{N}^+(U^n)$, but as remarked in Sec. 3 we have not been able to prove this, even for $n = 1$.

Another line of inquiry involves growth conditions other than bounded characteristic. For example, let

$$l(x) = \log(1 + x),$$
$$M(r, f) = \max_{|z| = r} |f(z)|$$

and consider the algebra E_Φ of entire functions of finite order, with metric induced by the subadditive functional

$$\| f \| = \sup_{r \geqslant 2} l(l(M(r, f)))/\log r. \tag{5.2}$$

It is easy to see that E_Φ is a complete additive topological group under this metric. Here the role of N^+ is taken over by the space E_0 of entire functions of order zero: E_0 is a complete linear topological space—in fact it is even locally convex.

THEOREM 5.3. *E_0 is the component of the origin in E_Φ.*

Proof. Let $\rho(f)$ denote the order of f, so

$$\rho(f) = \limsup_{r\to\infty} l(l(M(r,f)))/\log r.$$

Clearly ρ is subadditive and continuous on E_Φ, has the non-archimedian property (2.4), and annihilates E_0. It follows as in Sec. 2 that each $f \in E_\Phi$ of order >0 lies outside the component of the origin. Since E_0 is connected, it must therefore be precisely this component.

For $0 < \rho < \infty$ a similar result holds for the space E_ρ of all entire functions which are of growth (ρ, τ) for some $\tau < \infty$; that is, of order $\leqslant \rho$, and if of order ρ, then of type $\leqslant \tau$. Here the metric is induced by the functional

$$\|f\| = \sup_{r>1} l(M(r,f))/r^\rho,$$

and the subspace $E_{\rho,0}$ consisting of entire functions of growth $(\rho, 0)$ is a closed subspace which is a locally convex topological vector space. The functional

$$\tau(f) = \limsup_{r\to\infty} r^{-\rho} \log M(r,f)$$

is continuous, subadditive, non-archimedian, and vanishes precisely on $E_{\rho,0}$; hence $E_{\rho,0}$ is the component of the origin in E_ρ.

It might be of interest to investigate similar questions for other growth classes of entire functions and functions analytic in the unit disc.

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