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FOURIER SERIES, MEAN LIPSCHITZ SPACES, AND BOUNDED MEAN OSCILLATION

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ABSTRACT. Using simple and direct arguments, we: (i) prove, without recourse to duality, that the mean Lipschitz spaces $\Lambda(p,1/p)$ are contained in BMO, and (ii) improve the Hardy-Littlewood $\Lambda(p,1/p)$ Tauberian theorem. Along the way we connect the Hardy-Littlewood result with a recent Tauberian theorem for BMO functions due to Ramey and Ullrich, give an exposition of the relevant classical properties of Mean Lipschitz spaces; and survey some known function theoretic applications of the spaces $\Lambda(p,1/p)$.

INTRODUCTION. We work mostly on the unit circle T, and study for $1 < p < \infty$ the spaces $\Lambda(p,1/p)$ consisting of functions $f \in L^p(T)$ for which $\|f - f_t\|_p = O(t^{1/p})$ as $t \to 0$, where $f_t(x) = f(x - t)$. These spaces increase with p, and while none of them consists entirely of bounded functions ($\log(1-e^{ix})$ belongs to all of them), they all lie "on the border of continuity." More precisely, if in the definition of $\Lambda(p,1/p)$ the exponent $1/p$ is replaced by anything larger, there results a space of functions, each of which, after possible correction on a set of measure zero, is continuous. Our interest in $\Lambda(p,1/p)$ derives from four sources:

(i) the observation of Cima and Petersen [5] that $\Lambda(2,1/2)$ lies inside BMO, the space of functions of bounded mean oscillation on T;

(ii) the fact that the essential range of a function of vanishing mean oscillation must be connected [26];

(iii) a 1928 result of Hardy and Littlewood which states that $\Lambda(p,1/p)$ is a Tauberian condition relating Cesàro and ordinary summability. More precisely, the Fourier series of $f \in \Lambda(p,1/p)$, if

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(C,1) summable at a point of T, must actually converge there ([15], Theorem 1, page 613);

(iv) a recent theorem of Ramey and Ullrich [23] asserting that BMO is a Tauberian condition relating Abel summability and differentiability of indefinite integrals.

The original proofs of (i) and (iii) require complicated preliminaries. That of (i) in [5] uses the deepest part of the Fefferman-Stein duality theorem: the characterization of BMO functions by Carleson measures ([14], Chapter 6, Theorem 3.4, page 240; [10]); while Hardy and Littlewood prove (iii) by interpolating between ordinary Cesàro summability, and Cesàro boundedness of negative orders. Mysteriously, neither Hardy and Littlewood nor their successors in the literature appear to have considered the question of whether Abel summability at a point implies summability for the Fourier series of a $\Lambda(p,1/p)$ function.

This paper addresses both these complaints, and treats some additional topics suggested by the connection between the spaces $\Lambda(p,1/p)$ and BMO. First, we give a direct proof that $\Lambda(p,1/p) \subset$ BMO for all finite p. There is a corresponding containment between the "little oh" space $\lambda(p,1/p)$ and VMO, the space of functions of vanishing mean oscillation, which along with (ii) above shows that functions in $\lambda(p,1/p)$ must have connected essential range. We then extend the Hardy-Littlewood $\Lambda(p,1/p)$ Tauberian theorem to Abel summability by proving that at each point of the circle the sequence of Fourier partial sums of each $\Lambda(p,1/p)$ function is slowly oscillating. The fact that Abel summability of the Fourier series implies summability then follows from a standard Tauberian theorem. Our argument is considerably simpler than that of Hardy and Littlewood in that it avoids interpolation arguments and negative order Cesàro means. We discuss the connection between the Hardy-Littlewood Tauberian theorem and the Ramey-Ullrich theorem mentioned in (iv) above.

Both Hardy and Littlewood ([15], Lemma 12, page 620) and Cima and Petersen ([5], Theorem 2.1 and Cor. 2.2) noted that a function belongs to $\Lambda(2,1/2)$ whenever its Fourier coefficients decay like $O(1/n)$. Thus our $\Lambda(p,1/p)$ Tauberian theorem can be viewed as an extension of Littlewood's $O(1/n)$ Tauberian theorem. By duality, the $O(1/n)$ sufficient condition for membership in $\Lambda(2,1/2)$ can also be regarded as a generalization of Hardy's inequality from functions in the Hardy space H^1 to functions in a somewhat larger space.

Several classical papers extend the work of Hardy and Littlewood on Fourier series of mean Lipschitz functions (see [11] for further references, and for a unified treatment of some of this) and there is a vast literature about various generalizations of these spaces (see [4], [7], [12], [13], [17], [22], [25], [31] - [33] for the flavor of some of these generalizations, and for further references). Thus it seems quite possible that, while we have not yet come across them, our "new" results may already be known. However, much of the literature of mean Lipschitz spaces deals with settings far more complicated than ours, and is therefore not always as accessible as it should be to researchers in one variable function theory. For this reason we feel that an account of our work placed within the context of a detailed discussion of the relevant classical properties of mean Lipschitz spaces may, in any case, be of interest to function theorists.

Accordingly, we adhere to the following ground rules. We state without proof, but with detailed references: (i) function theoretic facts that can be found in Duren's book [8] on H^p theory, or Rudin's text [24]; (ii) basic facts about BMO such as are set out in Garnett's book [14], and (iii) classical Tauberian theorems for numerical series. On the other hand we give detailed proofs of all prerequisites on mean Lipschitz spaces that do not occur in these sources.

Here is an outline of the rest of the paper. Section 1 contains definitions, notation, and first properties of the mean Lipschitz spaces. Here we review the characterization of these spaces via Poisson integrals, and their resulting self-conjugacy. In the second section we discuss containments among the mean Lipschitz spaces, and prove that $\Lambda(p,1/p) \subset BMO$. The proof of our Tauberian theorem, as well as the discussion of the Ramey-Ullrich theorem, occupies section 4. This proof depends on a well-known characterization, presented in section 3, of $\Lambda(p,1/p)$ by the degree to which its members can be approximated by Fourier partial sums. The approximation theorem follows from a Littlewood-Paley type dyadic decomposition theorem, as used in [27]. The dyadic decomposition leads to a characterization of the Fourier coefficients of functions in $\Lambda(2,1/2)$, which in turn yields the previously mentioned "$O(1/n)$" sufficient condition for membership in $\Lambda(2,1/2)$. In the fifth and final section we comment how this sufficient condition generalizes Hardy's inequality for H^1, and survey a few other situations in function theory where the spaces $\Lambda(p,1/p)$ occur.

1. PRELIMINARIES. $L^p(T)$ $(1 \le p < \infty)$ denotes the space of (equivalence classes of) 2π-periodic measurable functions f on the real line for which

$$\|f\|_p^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p dt$$

while $L^\infty(T)$ is the corresponding space of essentially bounded measurable functions. The translate f_t of a function f on the real line by the real number t is defined by: $f_t(x) = f(x - t)$ for all real numbers x.

1.1. MEAN LIPSCHITZ SPACES $\Lambda(p,\alpha)$. For $1 \le p \le \infty$ and $0 < \alpha \le 1$ we define $\Lambda(p,\alpha)$ to be the collection of $f \in L^p(T)$ for which there exists a constant $C < \infty$ such that:

$$\|f_t - f\|_p \le C|t|^\alpha \quad \text{for all } t \in [-\pi,\pi].$$

If $p = \infty$ we write Λ_α instead of $\Lambda(\infty,\alpha)$. This is the usual Lipschitz space for the exponent α. More precisely, $f \in \Lambda_\alpha$ if and only if f coincides a.e. with a 2π periodic function F for which:

$$|F(x-t) - F(x)| \le |t|^\alpha \quad \text{for all } x, t \in [-\pi,\pi].$$

It is not difficult to show that the norm

$$\|f\|_{p,\alpha} = \|f\|_p + \sup\{t^{-\alpha}\|f_t - f\|_p : t \in [-\pi,\pi]\}$$

turns $\Lambda(p,\alpha)$ into a Banach space.

Clearly the spaces $\Lambda(p,\alpha)$ decrease as either p or α increases (with the other index held fixed). In the next section we examine the containments between these spaces in more detail. The key to these results, as well as to much of our subsequent work, is a useful characterization of mean Lipschitz spaces in terms of Poisson integrals.

1.2. POISSON INTEGRALS. Let U denote the open unit disc of the complex plane. For $f \in L^p(T)$, we denote by P[f] the Poisson integral of f:

$$P[f](re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, f(t)\, dt\,,$$

where $P_r(t) = \mathrm{Re}\{(1+re^{it})/(1-re^{it})\}$, and $re^{i\theta} \in U$. It is well known that $u = P[f]$ is harmonic in U, that the integral means

$$M_p(u,r) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{it})|^p dt\right\}^{1/p}$$

are bounded, and that the radial limit

$$u^*(e^{i\theta}) = \lim u(re^{i\theta}) \quad (r \to 1-)$$

exists and equals $f(\theta)$ for a.e. real θ ([8], Chapter 1). Conversely, if u is any harmonic function on U for which the integral means $M_p(u,r)$ are bounded, then $f = u^*$ exists a.e., belongs to $L^p(T)$, and $u = P[f]$. In short, if h^p denotes the class of harmonic functions u for which $M_p(u,r)$ is bounded, then the radial limit map $u \to u^*$ establishes an isometric isomorphism between h^p, taken in the natural norm imposed by its definition, and $L^p(T)$. The next result identifies the class of harmonic functions that corresponds in the same manner with $\Lambda(p,\alpha)$.

1.3. PROPOSITION. Suppose u is harmonic in U, $u = \mathrm{Re}\, F$ where F is holomorphic in U, $1 \le p \le \infty$, and $0 < \alpha < 1$. Then following conditions are equivalent:

(a) $u = P[f]$ for some $f \in \Lambda(p,\alpha)$,

(b) $M_p(F',r) \le C(1-r)^{\alpha-1}$ for all $0 \le r < 1$,

When (b) holds, the functional

$$f \to |u(0)| + \sup \{M_p(F',r)(1-r)^{1-\alpha}: 0 \le r < 1\}$$

is a norm on $\Lambda(p,\alpha)$ that is equivalent to $\|\cdot\|_{p,\alpha}$.

With F in place of u, this result was proved by Hardy and Littlewood ([15], Theorem 3, page 625). It can also be found in [8] (Theorem 5.4, page 78). The proof is exactly the one given in these references, except that in proving (a) → (b) one represents F by a completed Poisson integral of f, rather than a Cauchy integral.

It follows easily from Proposition 1.3 that, as we mentioned in the Introduction, the function $\log|1 - e^{i\theta}|$ is in $\Lambda(p,1/p)$.

Higher dimensional versions of this result, for more general spaces, can be found in [28], Chapter 5, and [31].

It follows from the M. Riesz theorem that the classes $\Lambda(p,\alpha)$ are self-conjugate if $1 < p < \infty$. However the result above gives a more elementary proof, valid even if $p = 1$ or ∞.

1.4. COROLLARY. If $1 \le p \le \infty$, $0 < \alpha < 1$, and $f \in \Lambda(p,\alpha)$, then so is its conjugate function $\tilde{f}$.

In [15] Hardy and Littlewood also prove Corollary 1.4 for the case $p > 1$, $0 < \alpha \le 1$; and the case $p = 1$, $0 < \alpha < 1$ (Lemma 13, page 621). The result is false for $p = \alpha = 1$. Indeed, $f \in \Lambda(1,1)$ if and only if f coincides a.e. with a function of bounded variation ([15], Lemma 9, page 619), and the class of functions of bounded variation is not self-conjugate.

2. CONTAINMENTS AMONG THE SPACES $\Lambda(p,\alpha)$.

We observed in the last section that the spaces $\Lambda(p,\alpha)$ decrease as either p or α increases. The next result gives more precise information.

2.1 PROPOSITION ([15] Theorem 5, p. 627). Suppose $1 \le p \le q \le \infty$, $0 < \alpha < 1$, and $\delta = p^{-1} - q^{-1}$. Then $\Lambda(p,\alpha) \subset \Lambda(q,\alpha-\delta)$.

PROOF. By the self-conjugacy of the spaces in question, it is enough to prove the theorem for $f \in \Lambda(p,\alpha)$ with Fourier transform vanishing on the negative integers. Then $F = P[f]$ is holomorphic in U, and by Proposition 1.3:

$$M_p(F',r) \le C_p \|f\|_{p,\alpha}(1-r)^{\alpha-1} \qquad (0 < r < 1).$$

The Hardy-Littlewood theorem on comparative growth of means ([15], Theorem 2, page 623; [8], Theorem 5.9, page 84) asserts that if $g \in H^p$, with p,q, and δ as above, then

$$M_q(g,r) \le C_{p,q}\|g\|_p (1-r)^{-\delta}.$$

We apply this result to the dilated function $g = (F')_r$ to obtain

$$M_q(F',r^2) \le C_{p,q} M_p(F',r)(1-r)^{-\delta}.$$

Thus:

$$M_q(F',r^2) \le C_{p,q}\|f\|_{p,\alpha}(1-r)^{\alpha-\delta-1},$$

which, along with another application of Proposition 1.3, proves the desired result. ///

Upon respectively setting $\alpha = 1/p$ and $q = \infty$, we justify the claims made in the Introduction that the spaces $\Lambda(p,1/p)$ increase with p, and lie on the border of continuity.

2.3. COROLLARY ([15], Theorems 5 and 6, pp. 627-8).

(a) If $1 < p < q < \infty$, then $\Lambda(p,1/p) \subset \Lambda(q,1/q)$.

(b) If $\alpha > 1/p$, then $\Lambda(p,\alpha) \subset \Lambda_{\alpha-(1/p)}$, hence each $f \in \Lambda(p,\alpha)$ coincides a.e. with a continuous function.

2.4. BOUNDED MEAN OSCILLATION. Suppose $f \in L^1(T)$. If I is a subinterval of $[0,2\pi]$, let $|I|$ denote its length, and write

$$f_I = \frac{1}{|I|}\int_I f(t)\,dt \quad .$$

Set

$$\|f\|_* = \sup\left\{\frac{1}{|I|}\int_I |f(t)-f_I|\,dt : I \text{ a subinterval of } [0,2\pi]\right\}.$$

The space BMO of functions of *bounded mean oscillation* is the collection of $f \in L^1(T)$ for which $\|f\|_* < \infty$. The John-Nirenberg theorem ([18]; [14], Chapter VI, Theorem 2, page 230) implies that $BMO \subset L^p(T)$ for all $p < \infty$, and that the same space, with equivalent norm, results if we redefine $\|f\|_*$ by:

$$\|f\|_*^p = \sup\left\{\frac{1}{|I|}\int_I |f(t)-f_I|^p\,dt : I \text{ a subinterval of } [0,2\pi]\right\}.$$

As we mentioned in the Introduction, Cima and Petersen [5] used deep results about BMO to show that $\Lambda(2,1/2) \subset BMO$. Here is a generalization (since the spaces $\Lambda(p,1/p)$ increase with p), for which we give a direct proof.

2.5. THEOREM. For $1 \le p < \infty$, $\Lambda(p,1/p) \subset BMO$.

PROOF. By the translation-invariance of both spaces, it is enough to show that for each $f \in \Lambda(p,1/p)$ there exists $C < \infty$ such that for each $0 < \delta < \pi/2$, upon letting $I = [-\delta,\delta]$ we have:

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} |f(t)-f_I|^p\,dt \le C\ .$$

Suppose $f \in \Lambda(p,1/p)$ with $\|f\|_{p,1/p} \le 1$, and fix $0 < \delta < \pi/2$. Then for all real t:

$$2\pi t \ge \int_{-\pi}^{\pi} |f(s-t) - f(s)|^p\,ds\ .$$

We integrate both sides of this inequality over the interval $[-2\delta,2\delta]$; then successively use Fubini's theorem and change variables on the resulting inner integral to obtain:

$$8\pi\delta^2 \ge \int_{-2\delta}^{2\delta}\int_{-\pi}^{\pi} |f(s-t) - f(s)|^p\,ds\,dt$$

$$= \int_{-\pi}^{\pi}\int_{-2\delta}^{2\delta} |f(s-t) - f(s)|^p\,dt\,ds$$

$$= \int_{-\pi}^{\pi}\int_{s-2\delta}^{s+2\delta} |f(t) - f(s)|^p\,dt\,ds$$

$$\geq \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} |f(t) - f(s)|^p \, dt \, ds$$

where the last line follows from the fact that if $-\delta \leq s,t \leq \delta$, then $s - 2\delta \leq t \leq s + 2\delta$. Now divide both sides of the above inequality by $4\delta^2$ and apply Jensen's convexity theorem:

$$2\pi \geq \frac{1}{2\delta} \int_{-\delta}^{\delta} \left[\frac{1}{2\delta} \int_{-\delta}^{\delta} |f(t) - f(s)|^p \, dt \right] ds$$

$$\geq \frac{1}{2\delta} \int_{-\delta}^{\delta} \left| \frac{1}{2\delta} \int_{-\delta}^{\delta} [f(t) - f(s)] \, dt \right|^p ds$$

$$= \frac{1}{2\delta} \int_{-\delta}^{\delta} | f(s) - f_I |^p \, ds \,,$$

where $I = [-\delta, \delta]$. Since $0 < \delta < \pi/2$ is arbitrary, we have achieved the desired result. ///

2.6. "LITTLE OH." The collection $\lambda(p,\alpha)$ of $f \in L^p(T)$ for which $\|f - f_t\|/t^\alpha \to 0$ as $|t| \to 0$ is a closed subspace of $\Lambda(p,\alpha)$, and the previously stated results all have "little oh" analogues with $\lambda(p,\alpha)$ replacing $\Lambda(p,\alpha)$. In particular, $\lambda(p,1/p) \subset$ VMO, the space of functions of vanishing mean oscillation. It has recently been shown that every VMO function has connected essential range [26], hence,

PROPOSITION. If $1 < p < \infty$, and $f \in \lambda(p,1/p)$, then the essential range of f is connected.

2.7. GENERALIZED MEAN LIPSCHITZ SPACES. The following generalization of the spaces $\Lambda(p,\alpha)$ occurs frequently in the literature ([31] - [Ta3]; [28] Chapter V, section 5; [12]). If $1 < p, q < \infty$, and $0 < \alpha < 1$, say a function $f \in L^p(T)$ belongs to $\Lambda(p,q,\alpha)$ if

$$\int_0^\infty \|f - f_t\|_p^q \, t^{-1-\alpha q} \, dt < \infty$$

(note that the convergence of the integral depends only upon the behavior of the integrand for t near 0). Our spaces $\Lambda(p,\alpha)$ correspond to the limiting case $q = \infty$ here. These generalized mean Lipschitz spaces play no role in this paper because of the containment: $\Lambda(p,q,\alpha) \subset \lambda(p,\alpha)$ ([12], page 125).

3. PARTIAL SUMS AND DYADIC BLOCKS. For $f \in L^1(T)$ and n an integer, let $\hat{f}(n)$ denote the nth Fourier coefficient of f:

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{int} \, dt \,.$$

For n positive, write

$$s_n f(\theta) = \sum_{|k| \leq n} \hat{f}(k) e^{ik\theta}$$

for the nth partial sum of the Fourier series of f, and setting

$$I(n) = \{k \text{ an integer: } 2^n < |k| \leq 2^{n+1}\},$$

write

$$\Delta_n f(\theta) = \sum_{k \in I(n)} \hat{f}(k) e^{ik\theta} \,,$$

for the nth dyadic block of that series. A standard argument involving the M. Riesz theorem shows that for each $1 < p < \infty$ there exists a constant $C_p < \infty$ for which:

$$\|s_n f\|_p \le C_p \|f\|_p, \quad \text{and} \quad \|\Delta_n f\|_p \le C_p \|f\|_p,$$

for all $f \in L^p(T)$, and all positive integers n. The first of these inequalities, along with the density of trigonometric polynomials in $L^p(T)$, shows that $\|s_n f - f\|_p \to 0$ for each $f \in L^p(T)$ $(1 < p < \infty)$. These matters are discussed, for example, in [24], Ch. 17, Problem 25.

In this section we survey some known results which relate the mean smoothness of f with the speed at which it is approached by $s_n f$. The main result is:

3.1. THEOREM. Suppose $1 \le p < \infty$ and $0 < \alpha < 1$. Then for $f \in L^p(T)$, the following three conditions are equivalent.

(a) $f \in \Lambda(p,\alpha)$.

(b) $\|\Delta_n f\|_p = O(2^{-n\alpha})$ as $n \to \infty$.

(c) $\|f - s_n f\|_p = O(n^{-\alpha})$ as $n \to \infty$.

The equivalence of (a) and (c) is mentioned without proof in Hardy and Littlewood's paper [15]. That of (a) and (b) often serves as the basis for the definition of spaces generalizing mean Lipschitz spaces in various settings (see [4], [13], [27], for example). In the next section, Theorem 3.1 will play an important role in the proof of our $\Lambda(p,1/p)$ Tauberian theorem. It also has notable appeal in the case $p = 2$, where through the Parseval identity it characterizes the Fourier coefficients of functions in $\Lambda(2,1/2)$.

3.2. COROLLARY. Suppose $f \in L^2(T)$ and $0 < \alpha < 1$. Then the following three conditions on f are equivalent:

(a) $f \in \Lambda(2,\alpha)$,

(b) $\displaystyle\sum_{k \in I(n)} |\hat{f}(k)|^2 = O(2^{-2n\alpha})$ as $n \to \infty$,

(c) $\displaystyle\sum_{|k| \ge n} |\hat{f}(k)|^2 = O(n^{-2\alpha})$ as $n \to \infty$.

3.3. COROLLARY (see [15], Lemma 12; and [5], Corollary 2.2). If $f \in L^1(T)$ and $|\hat{f}(n)| = O(1/|n|)$ as $n \to \infty$, then $f \in \Lambda(2,1/2)$.

The proof of Theorem 3.1 requires some preliminary estimates. If f is a trigonometric polynomial:

$$f(\theta) = \sum a_n e^{in\theta} \quad \text{(finite sum)},$$

let us write

$$f'(\theta) = \sum n a_n e^{i(n-1)\theta},$$

and for $r \ge 0$,

$$f_r(\theta) = \sum a_n r^{|n|} e^{in\theta}.$$

The lemma below generalizes to $1 \le p \le \infty$ some estimates that are obvious if $p = 2$.

3.4. LEMMA. Suppose $0 \le N \le M < \infty$, and

$$f(\theta) = a_N e^{iN\theta} + \dots + a_M e^{iM\theta}.$$

Then for $1 \le p \le \infty$:

(a) $r^M \|f\|_p \le \|f_r\|_p \le r^N \|f\|_p \quad (0 \le r \le 1)$,

and

(b) $N\|f\|_p \le \|f'\|_p \le M\|f\|_p$.

PROOF. Both inequalities follow from a standard fact about convolutions: $\|K*f\|_p \leq \|K\|_1\|f\|_p$ whenever $K \in L^1(T)$ and $f \in L^p(T)$. To prove (a), observe first that for any trigonometric polynomial f, if $0 \leq r < 1$, then $f_r = P_r*f$, where P_r is the Poisson kernel. Now for f as in the hypothesis of the lemma, let $h(\theta) = e^{-iN\theta}f(\theta)$ and $g(\theta) = e^{-iM\theta}f_r(\theta)$. It is easy to check that:

(1) $f_r(\theta) = r^N e^{iN\theta}h_r(\theta)$,

and

(2) $r^M f(\theta) = e^{iM\theta}g_r(-\theta)$.

From (1):

$$\|f_r\|_p = r^N\|h_r\|_p = r^N\|P_r*h\|_p \leq r^N\|P_r\|_1\|h\|_p = r^N\|f\|_p$$

This proves the second inequality of (a). For the first one we use (2) above, along with the same convolution inequality to obtain:

$$r^M\|f\|_p = \|g_r\|_p \leq \|g\|_p = \|f_r\|_p \ ,$$

which completes the proof of (a).

Part (b) is proved similarly. Let $g(\theta) = e^{-i(N-1)\theta}f'(\theta)$, and

$$K(\theta) = \lim_{n\to\infty}\sum_{k=-n}^{n}\frac{1}{|k|+N}e^{ik\theta} \ .$$

Since the coefficient sequence for K is symmetric about 0, and convex for non-negative n, the series on the right converges whenever $e^{i\theta} \neq 1$; moreover the resulting function is positive and integrable on T ([8], Theorem 4.5, page 64). In particular:

$$\|K\|_1 = \hat{K}(0) = 1/N.$$

Now $e^{-iN\theta}f(\theta) = K*g(\theta)$, so

$$\|f\|_p \leq \|K\|_1\|g\|_p = N^{-1}\|f'\|_p,$$

which proves the first inequality of (b).

For the second inequality, write:

$$g(\theta) = Ma_M + (M-1)a_{M-1}e^{i\theta} + \ldots + Na_N e^{i(M-N)\theta} \ ,$$

and

$$K(\theta) = \sum_{|n|< M}(M - |n|)e^{in\theta} \ .$$

Then $g = K*h$, where $h(\theta) = e^{-iM\theta}f(-\theta)$. Thus:

$$\|f'\|_p = \|g\|_p = \|K*h\|_p \leq \|K\|_1\|h\|_p = \|K\|_1\|f\|_p \ ,$$

and the proof is completed by observing that K is just M times the M-1st Fejer kernel, so $\|K\|_1 = M$. ///

The example $f(\theta) = e^{iN\theta}$ shows that the inequalities of Lemma 3.4 cannot be improved. The second inequality in part (a) is a version of Bernstein's inequality ([19], page 17, Problem 12).

3.5. PROOF OF THEOREM 3.1. First some notation. If F is holomorphic in U, let s_nF and Δ_nF denote respectively the nth partial sum and nth dyadic block of the Taylor expansion of F about the origin. Since the spaces $\Lambda(p,\alpha)$ are self-conjugate for the indices considered, it suffices to prove the theorem for functions f whose Fourier transform vanishes on the negative integers, i.e. for f of "power series type". In the arguments below, "C" always denotes a finite positive constant which may vary from line to line, but never depends on the parameters n or r.

(1) → (2). Suppose $f \in \Lambda(p,\alpha)$ is of power series type, so its Poisson integral $F = P[f]$ is holomorphic on U. Fix a positive integer n. Then for $0 \leq r < 1$ we have:

$$M_p(\Delta_n(zF'),r) \ \leq \ C\,M_p(zF'),r) \ \leq \ C(1 - r)^{\alpha-1}$$

where the first inequality follows from the M. Riesz theorem (since $1 < p < \infty$), and the second from Proposition 1.3, the characterization of mean Lipschitz spaces by Poisson integrals. Now set $r = 1-2^{-n}$, and use successively both left-hand inequalities in Lemma 3.4:

$$\begin{aligned} 2^{n(1-\alpha)} &\geq C\, M_p(\Delta_n(zF'),r) \\ &\geq C r^{2^{n+1}} \|(\Delta_n f)'\|_p \qquad \text{(notation as in Lemma 3.4)} \\ &\geq C\, 2^n \|\Delta_n f\|_p . \end{aligned}$$

Thus $\|\Delta_n f\|_p \leq C^{-1} 2^{-n\alpha}$, as desired.

(2) → (3). Suppose $f \in L^p(T)$ obeys (2), and n is a fixed positive integer. Choose $N = 2^{j-1}$ so that $N < n \leq 2N$. By the M. Riesz theorem,

$$\begin{aligned} \|f - s_n f\|_p &\leq C \|f - s_N f\|_p \\ &\leq C \sum_{k \geq j-1} \|\Delta_k f\|_p \\ &\leq C \sum_{k \geq j-1} 2^{-k\alpha} \\ &\leq C\, 2^{-j\alpha} \\ &\leq C n^{-\alpha}, \end{aligned}$$

which is (3).

(3) → (1). Suppose $f \in L^p(T)$ satisfies (3), i.e.

$$\|s_n f - f\|_p \leq Cn^{-\alpha} \qquad (n = 1,2,\ldots).$$

Then by the M. Riesz theorem:

$$\|\Delta_n f\|_p \leq C\, 2^{-n\alpha}$$

for each positive integer n, so using both right-hand inequalities in Lemma 3.4, we obtain:

$$\begin{aligned} M_p(zF',r) &\leq \sum M_p(\Delta_n(zF'),r) \\ &\leq \sum r^{2^n} \|(\Delta_n f)'\|_p \qquad \text{(by right-hand inequality of 3.4(a)} \\ &\leq \sum r^{2^n} 2^{n+1} \|\Delta_n f\|_p \qquad (\text{" " " " } 3.4(b) \\ &\leq C \sum r^{2^n} 2^{n(1-\alpha)} , \end{aligned}$$

where in each line the range of summation is $0 \leq n < \infty$. Since

$$r^{2^n} 2^{n(1-\alpha)} \leq \sum_{k=2^{n-1}+1}^{2^n} r^k k^{-\alpha},$$

the previous estimate gives

$$M_p(F',r) \leq C \sum k^{-\alpha} r^k \leq C/(1-r)^{1-\alpha},$$

which, by Proposition 3.1, shows that $f \in \Lambda(p,\alpha)$, and completes the proof of the Theorem. ///

4. TAUBERIAN NATURE OF $\Lambda(p,1/p)$. Here we give our simple proof of the fact that the Fourier series of a function $f \in \Lambda(p,1/p)$ converges at a point whenever it is Abel summable there. As noted in the Introduction, $\Lambda(1,1)$ is essentially the space of functions of bounded variation, so we always assume without further mention that $1 < p < \infty$. We begin with a review of some summability matters, and hope the reader will not be offended that we begin at the beginning.

4.1. SUMMABILITY. Let $(a_n: 0 \leq n < \infty)$ be a sequence of complex numbers. The series $\sum a_n$ is said to be:

(i) *summable (to S)* if the sequence of partial sums

$$s_n = a_0 + a_1 + \ldots + a_n$$

converges (to S);

(ii) *Cesàro summable (to S)* if the sequence of arithmetic means

$$\sigma_n = (s_0 + s_1 + \ldots + s_n)/(n+1)$$

converges (to S); and

(iii) *Abel summable (to S)* if

$$\lim_{r \to 1-} \sum_{n=0}^{\infty} a_n r^n = S ,$$

where the tacit assumption is, of course, that the series on the left converges for all $0 \le r < 1$.

If $f \in L^1(T)$ and θ is real, then we say the Fourier series of f converges in one of these modes to S if the corresponding numerical series $\sum a_n$ does, where $a_0 = \hat{f}(0)$, and

$$a_n = \hat{f}(-n)e^{-in\theta} + \hat{f}(n)e^{in\theta} \qquad (n > 0).$$

In particular, the Fourier series of f converges at θ if and only if the sequence $(s_n f(\theta): n \ge 0)$ of symmetric partial sums defined in section 3 converges, and it is Abel summable at θ if and only if

$$\sum \hat{f}(n) r^{|n|} e^{in\theta} = P[f](re^{i\theta})$$

converges as $r \to 1-$.

As everyone knows: summability $\Rightarrow$ Cesàro summability $\Rightarrow$ Abel summability, but in general, not conversely. Here is the main result of this section.

4.2. $\Lambda(p,1/p)$ TAUBERIAN THEOREM. If the Fourier series of $f \in \Lambda(p,1/p)$ is Abel summable at θ, then it is summable at θ.

As we mentioned in the Introduction, Hardy and Littlewood proved this result under the stronger hypothesis that the Fourier series be Cesàro summable. Their proof, which involved interpolation between ordinary Cesàro summability and negative order Cesàro boundedness, is considerably more complicated than the one we will present for the stronger result stated above.

However their proof does give additional information: it implies that the Fourier series is Cesàro convergent for certain negative orders.

The crucial Tauberian concept in our proof is that of *slow oscillation.*

4.3. DEFINITION. A sequence $(a_n: n \ge 0)$ is said to be *slowly oscillating* if for every $\varepsilon > 0$ there exists $\lambda = \lambda(\varepsilon) > 1$ and a positive integer $N = N(\varepsilon)$ such that :

$$\max\{|a_j - a_k|: \ n \le j, k \le \lambda n\} < \varepsilon$$

whenever $n \ge N$.

For example, if $|a_n| = O(1/n)$, then the corresponding sequence of partial sums is slowly oscillating. This shows that the next result is a generalization of Littlewood's $O(1/n)$ Tauberian theorem.

4.4. SASZ'S TAUBERIAN THEOREM. Suppose (a_n) is a numerical sequence whose partial sums (s_n) form a slowly oscillating sequence. If the series $\sum a_n$ is Abel summable, then it is summable.

This result originally appeared in [29]. An eminently readable proof is presented in [30]. Theorem 4.2 follows immediately from Sasz's theorem and:

4.5. THEOREM. For every $f \in \Lambda(p,1/p)$ and real θ, the numerical sequence $(s_n f(\theta): n \ge 0)$ is slowly oscillating.

PROOF. According to the definition of slow oscillation, our goal is to prove that:

$$(1)\quad \overline{\lim_{n\to\infty}} \max_{n<m\le\lambda n} \left| s_m f(\theta) - s_n f(\theta) \right| \to 0 \text{ as } \lambda \to 1+ .$$

To this end, fix $n < m$ and let

$$Q_{nm}(\theta) = \sum_{n<k\le m} e^{ik\theta} .$$

Then:

$$(2)\quad s_n f(\theta) - s_m f(\theta) = Q_{nm} * f(\theta) = Q_{nm} * (f - s_n f)(\theta),$$

where the last equality follows from the fact that $s_n f * Q_{nm} \equiv 0$. Upon applying Holder's inequality and the translation invariance of Lebesgue measure to (2) we obtain:

$$(3)\quad \left| s_n f(\theta) - s_m f(\theta) \right| \le \| f - s_n f \|_p \| Q_{nm} \|_{p'}$$

where p' is the index conjugate to p. Now

$$\left| Q_{nm}(\theta) \right| \le 2 K_{m-n-1}(\theta),$$

where

$$K_m(\theta) = \left| 1 + e^{i\theta} + \ldots + e^{im\theta} \right|$$

$$= \left| \frac{\sin (m+1)(\theta/2)}{\sin (\theta/2)} \right|$$

Thus

$$\| Q_{nm} \|_{p'} \le 2 \| K_{m-n-1} \|_{p'} \le C (m - n)^{1/p} ,$$

where C is a constant that does not depend on m or n. The last inequality is a straightforward computation based on the closed-form expression for K_m and a change of variable. We leave the details to the reader. From inequality (3) above and Theorem 3.1:

$$\left| s_n f(\theta) - s_m f(\theta) \right| \le C (m - n)^{1/p} n^{-1/p} ,$$

so for $\lambda > 1$:

$$\sup_{n<m\le\lambda n} \left| s_m f(\theta) - s_n f(\theta) \right| \le C (\lambda - 1)^{1/p} .$$

This inequality yields (1), and completes the proof of the Theorem.

4.6. THE TAUBERIAN NATURE OF BMO. Recall from section 2 that $\Lambda(p, 1/p) \subset$ BMO. However BMO, or even VMO, cannot replace $\Lambda(p, 1/p)$ in Theorems 4.2 or 4.5, since there exist continuous functions on $\mathbf{T}$ whose Fourier series diverge at a given point. Nevertheless, Ramey and Ullrich [23] have recently shown that BMO is a Tauberian condition linking Abel and various other methods of summability. They work on the real line $\mathbf{R}$, instead of the circle. To state their result efficiently we need the notion of normalized dilate. If $K \in L^1(\mathbf{R})$ and $y > 0$, let $K_y(x) = y^{-1}K(x/y)$. Thus $K \in L^1(\mathbf{R})$, and $\| K_y \|_1 = \| K \|_1$. Ramey and Ullrich prove the following Tauberian theorem for BMO($\mathbf{R}$).

THEOREM ([23], Theorem 4.4). Suppose $f \in$ BMO($\mathbf{R}$), $x \in \mathbf{R}$, and $P[f](x,y) \to L$ as $y \to 0+$. Then also $f * K_y(x) \to L$ whenever $K \in L^1(\mathbf{R})$ obeys the following additional conditions:

(i) $\left| K(x) \right| <$ constant$(1 + x^2)^{-1}$ for all real x, and

(ii) $\displaystyle\int_{-\infty}^{\infty} K(x)\,dx = 1$.

The proof of this result involves an elegant mixture of functional analysis (weak* convergence in BMO($\mathbf{R}$)), and function theory (normal families), along with the crucial observation that the BMO(R) norm is dilation invariant. Actually Ramey and Ullrich work in higher dimensions, but state their result only for K the characteristic function of a ball. However their proof gives the full result stated above. Letting K be respectively the characteristic function of the interval $[-1/2, 1/2]$, and the Fejer (Cesàro) kernel:

$$K(x) = \frac{1}{\pi}\left[\frac{\sin x}{x}\right]^2,$$

this result yields:

COROLLARY. If the Fourier integral of $f \in BMO(R)$ is Abel summable to L at $x \in R$, then it is Cesàro summable to L at x, and also:

$$\lim_{h\to 0+} \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt = L .$$

This Corollary can be transferred to the unit circle by means of the Poisson summation formula, and the observation that if a 2π-periodic function is in BMO(T), then it is in BMO(R). Thus: *the Corollary above remains true if R is replaced by T and "Fourier integral" is replaced by "Fourier series".*

Since $\Lambda(p,1/p) \subset BMO(T)$, this result for the circle, along with Hardy and Littlewood's original Tauberian theorem for $\Lambda(p,1/p)$, give another proof of Theorem 4.2. In [15] (Theorem 1, page 613) Hardy and Littlewood also state that the convergence of the integral average in the above Corollary is a necessary and sufficient condition for the Fourier series of $f \in \Lambda(p,1/p)$ to converge at the point x. This also follows from the considerations above: If the averages converge to L, then as is well known ([24], Theorem 11.2, page 257), $P[f](re^{i\theta}) \to L$ as $r \to 1-$. Thus by Theorem 4.2, $s_nf(\theta) \to L$. Conversely, if $s_nf(\theta) \to L$, then since $\Lambda(p,1/p) \subset$ BMO, the circle version of the Ramey-Ullrich Corollary above asserts that the integral averages converge to L.

5. $\Lambda(p,1/p)$ AND FUNCTION THEORY. In this final section we survey some function theoretic situations in which the spaces $\Lambda(p,1/p)$ arise naturally.

5.1. HARDY'S INEQUALITY. That the coefficient condition $|\hat{f}(n)| = O(1/|n|)$ suffices for $f \in$ BMO seems to be part of the folklore. It is usually obtained by duality from Hardy's inequality:

$$\sum \frac{|\hat{f}(n)|}{n+1} \leq \pi \|f\|_1$$

for $f \in H^1$ (see [2], page 25, for example). However, following [5], Corollary 3.3 gives the result directly, and therefore by duality provides a different proof of Hardy's inequality (although with a less precise constant).

In fact, Corollary 3.3 gives a result more general than Hardy's inequality. Flett [12] has shown that for $1 < p < \infty$, $\Lambda(p,\alpha)$ is the dual space of the space $X(p',1-\alpha)$ consisting of functions u harmonic on U for which

$$\|u\|_X = \int_{-\pi}^{\pi} M_{p'}(u,r)(1-r)^{1-\alpha}dr < \infty ,$$

and the two spaces are paired by integration on the circle. Now the same duality argument that lead from Hardy's inequality to the $O(1/n)$ sufficient condition for BMO can be reversed to give a version of Hardy's inequality for $X(2,1/2)$. Since $\Lambda(2,1/2) \subset$ BMO, it follows easily that the reverse inequality is true of the preduals: $\mathrm{Real}\{H^1\} \subset X(2,1/2)$. More directly, this last inclusion follows from a result of Hardy and Littlewood ([8], Theorem 5.11, p. 87]). Thus, the $O(1/n)$ sufficient condition for membership in $\Lambda(2,1/2)$ leads to an improvement of Hardy's inequality.

5.2. BLASCHKE PRODUCTS. Suppose (z_n) is a sequence of points in the unit disc, arranged in order of increasing modulus. Let $d_n = 1 - |z_n|$, and suppose $\Sigma d_n < \infty$. Then the infinite product

$$B(z) = \prod_{n=0}^{\infty} \omega_n \frac{z_n - z}{1 - \bar{z}_n z}$$

where $\omega_n = \bar{z}_n / |z_n|$, converges uniformly on compact subsets of U to a bounded holomorphic function B, called *the Blaschke product with zeros* (z_n) . Moreover, B is an inner function, that is, its radial limit

$$B^*(e^{i\theta}) = \lim B(re^{i\theta}),$$

which exists a.e. by Fatou's theorem, has modulus 1 a.e (see [8], Chapter 2, or [24], Sections 15.21-15.24, pages 333-336).

Suppose $B(z) = \Sigma \hat{B}(n) z^n$ is the Taylor expansion of B about the origin. In 1962 D.J. Newman and H.S. Shapiro [21] proved a number of interesting results about the Taylor coefficients of Blaschke products, and more generally, of inner functions. In the first place, they showed among inner functions, only the Blaschke products with finitely many factors have Taylor coefficients which tend to zero like $o(1/n)$. This can be seen in modern terms as follows. It is known [Ste], [26] that among inner functions, only the finite Blaschke products can have boundary function belonging to VMO, and we know from section 2 that $\lambda(2,1/2) \subset$ VMO. Now the $o(1/n)$ coefficient condition for an inner function f implies that

$$(*) \qquad n \sum_{|k|>n} |\hat{f}(k)|^2 \to 0 \quad \text{as } n \to \infty,$$

which, by the "little oh" version of Corollary 3.2, is necessary and sufficient for $f^* \in \lambda(2,1/2)$. Note that this argument actually

improves the Newman-Shapiro result; it shows that any inner function whose Taylor coefficients obey (*) above must be a finite Blaschke product.

Newman and Shapiro also constructed Blaschke products B with coefficients $|\hat{B}(n)| = O(1/n)$. In fact they showed that any Blaschke product for which

$$(1) \qquad \sup_n d_{n+1}/d_n < 1$$

has this property. In our language: condition (1) implies $B^* \in \Lambda(2,1/2)$. In the converse direction, Ahern [1] showed in 1979 that (for Blaschke products) necessary and sufficient for

$$(2) \qquad d_n = O(a^n) \qquad (\text{some } 0 < a < 1)$$

is

$$(3) \qquad \sum_{k=0}^{N} \sum_{n=k}^{\infty} |\hat{B}(n)|^2 = O(\log N)$$

([A], Lemma 3.1 and Theorem 3.3, pages 327 - 331). Now if $B^* \in \Lambda(2,1/2)$, then by our Corollary 3.2, the estimate (3) above holds, hence the zeros of B must satisfy (2). Finally, in 1982 Verbitskii [34] obtained the complete result:

THEOREM. For B a Blaschke product, the following four conditions are equivalent:

(a) The zeros of B can be decomposed into finitely many sequences, each of which satisfies (1) above,

(b) $B^* \in \Lambda(2,1/2)$,

(c) $B^* \in \Lambda(p,1/p)$ for some $1 < p < \infty$,

(d) $|\hat{B}(n)| = O(1/n)$.

5.3. UNIVALENT FUNCTIONS. S is the class of analytic, univalent functions f on U for which $f(0) = 0$ and $f'(0) = 1$. The associated logarithmic function $g(z) = \log[f(z)/z]$ was shown by Baernstein [3] to belong to BMOA, the class of functions in H^2 with boundary function in BMO, and an alternate proof was later given by Cima and Schober [6]. Thus it is natural to ask if g must actually belong to $\Lambda(2,1/2)$. However Hayman [16] has constructed an example which shows that this is not the case. On the other hand, in [6] Cima and Schober show that it *is* the case if f is a support point of S. This raises the apparently open question, first asked by Allen Shields, of whether $g \in \Lambda(2,1/2)$ whenever f is an *extreme point* of S (for more details see [9]). Duren and Leung [9] have shown that $g \in \Lambda(2,1/2)$ whenever the modulus of f dominates a positive multiple of $(1 - |z|)^{-2}$ on some sequence that tends to the boundary.

Of course, smoothness classes of analytic functions show up in many other contexts. For example, applications to approximation theory and operator theory can be found in the appendix by Hruschev and Peller to Nikolskii's book [20].

BIBLIOGRAPHY

[1] P. Ahern, "The mean modulus and the derivative of an inner function", Indiana Univ. Math. J. 28 (1979), 311 - 347.

[2] A. Baernstein II, "Analytic functions of bounded mean oscillation," in *Aspects of Contemporary Complex Analysis*, Conference Proceedings, edited by D.A. Brannan and J.G. Clunie, Academic Press, New York, 1980.

[3] ———————, "Univalence and bounded mean oscillation," Michigan Math. J. **23** (1976), 217 - 223.

[4] J. Boman and H.S. Shapiro, "Comparison theorems for a generalized modulus of continuity," Arkiv för Mat. **9** (1971), 91-116.

[5] J.A. Cima and K.E. Petersen, "Some analytic functions whose boundary values have bounded mean oscillation," Math. Z. **147** (1976), 237-247.

[6] ——————— and G. Schober, "Analytic functions with bounded mean oscillation and logarithms of H^p functions," Math. Z. **151** (1976) 295 - 300.

[7] R.A. DeVore and R.C. Sharpley, "Maximal functions measuring smoothness," Memoirs Amer. Math. Soc. **47**, #293, (1984) 1-115.

[8] P.L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[9] ——————— and Y.J. Leung, "Logarithmic coefficients of univalent functions," J. d'Analyse Math. 36 (1979), 36 - 43.

[10] C. Fefferman and E.M. Stein, "H^p spaces of several variables," Acta Math. **129** (1972) 137-193.

[11] T.M. Flett, "Some more theorems concerning absolute summability of Fourier series and power series," Proc. London Math. Soc. (3) **8** (1958), 357-387.

[12] ________, "Lipschitz spaces of functions on the circle and the disc," J. Math. Anal. & App. **39** (1972), 125-158.

[13] M. Frazier and B. Jawerth, "Decomposition of Besov spaces," Indiana Univ. Math. J. **34** (1985), 777-799.

[14] J. Garnett, *Bounded Analytic Functions,* Academic Press, New York, 1981.

[15] G.H. Hardy and J.E. Littlewood, "A convergence criterion for Fourier series," Math. Z. **28** (1928), 612 - 634.

[16] W. Hayman, "The logarithmic derivative of multivalent functions," Michigan Math. J. **27** (1980), 149 - 179.

[17] S. Janson, "Generalization of Lipschitz spaces and an application to Hardy spaces and bounded mean oscillation," Duke Math. J. **47** (1980), 959-982.

[18] F. John and L. Nirenberg, "On functions of bounded mean oscillation," Comm. Pure Appl. Math. **14** (1961), 415-426.

[19] Y. Katznelson, *Introduction to Harmonic Analysis,* J. Wiley & Sons, New York, 1968.

[20] N.K. Nikolskii, Treatise on the Shift Operator. Springer-Verlag, New York, 1985.

[21] D.J. Newman and H.S. Shapiro, "The Taylor coefficients of inner functions," Michigan Math. J. **9** (1962), 249-255.

[22] J. Peetre, *New Thoughts on Besov Spaces,* Duke Univ. Math. Series **1**, Dept. of Math. Duke Univ. Durham, N.C. 1976.

[23] W. Ramey and D. Ullrich, "On the behavior of harmonic functions near a boundary point," preprint 1987.

[24] W. Rudin, *Real and Complex Analysis, 2nd ed.* McGraw Hill, New York, 1974.

[25] H. S. Shapiro, "A Tauberian theorem related to approximation theory," Acta Math. **120** (1968), 279 - 292.

[26] J. H. Shapiro, "Cluster set, essential range, and distance estimates in BMO," Michigan Math. J. **34** (1987), 323 - 336.

[27] W. T. Sledd, "Some results about spaces of analytic functions introduced by Hardy and Littlewood," J. London Math. Soc. (2) **9** (1974), 328 - 336.

[28] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions.* Princeton Univ. Press, Princeton, N.J., 1970.

[29] O. Szász, "Converse theorems of summability for Dirichlet series," Trans. Amer. Math. Soc. **39** (1936) 117 - 130.

[30] ________, *Introduction to the Theory of Divergent Series*, Lecture notes, Univ. of Cincinnatti, Cincinnatti, Ohio 1944.

[31] - [33] M. H. Taibleson, "On the theory of Lipschitz spaces of distributions on Euclidean n-space; I. Principal properties; II. Smoothness and integrability of Fourier transforms; III. Translation invariant operators, duality, and interpolation," J. Math. Mech. **13** (1964), 407 - 479; **14** (1965), 821 - 839; **15** (1966) 973 - 981.

[34] I.E. Verbitskii, "On Taylor coefficients and L^p moduli of continuity of Blaschke products", Zap. Nauchn. Sem. Leningrad. Otdel. Inst. Steklov. (LOMI) **107** (1982), 27 - 35 (Russian, English summary). MR 84d:30059.

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