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The essential norm of a composition operator

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Abstract

We express the essential norm of a composition operator on the Hardy space H^2 as the asymptotic upper bound of a quantity involving the Nevanlinna counting function of the inducing map. There results a complete function theoretic characterization of the compact composition operators on H^2. Similar results hold for the weighted Bergman spaces of the unit disc. As consequences we obtain:

(i) estimates of the essential norm of a composition operator in terms of the angular derivative of its inducing map;

(ii) a new proof of a recently obtained characterization of the compact composition operators on the weighted Bergman spaces; and

(iii) a new proof of a peak set theorem for holomorphic Lipschitz functions.

1. Introduction

Let U denote the unit disc of the complex plane and let φ be a holomorphic function on U with $\varphi(U) \subset U$. Then the equation $C_\varphi f = f \circ \varphi$ defines a *composition operator* C_φ on the space of holomorphic functions in U, and Littlewood's subordination principle ([5], [10]) assures that C_φ acts boundedly on the Hardy space H^2. The goal of this paper is to give a function theoretic characterization of those φ for which C_φ is *compact*. More generally, we are able to express the *essential norm* of C_φ (its distance, in the operator norm, from the space of compact operators on H^2) in terms of an asymptotic bound involving the Nevanlinna counting function of φ: it is the corresponding "little oh" condition which characterizes the compacts (Theorem 2.3).

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This result completes a line of investigation begun in 1969 by H. J. Schwartz [16], and continued by the author in collaboration with P. D. Taylor [18], and B. D. MacCluer [12]. Throughout, the goal has been to make quantitative sense out of a single intuitive principle: *C_φ is compact on H^2 if and only if the values of φ are not too often too close to the boundary of U.*

This principle is best understood by starting at its extremes. Consider first the map $\varphi(z) \equiv z$, whose values are, by the Schwarz lemma, as close as possible to the boundary. In this case C_φ, being the identity map on H^2, is clearly not compact. At the other extreme lie the maps φ whose values never approach the boundary, that is, the ones for which

$$\|\varphi\|_\infty = \sup\{|\varphi(z)|: z \in U\} < 1.$$

A straightforward argument, based on the fact that the unit ball of H^2 is a normal family, shows that the composition operators induced by such maps are all compact.

It is the intermediate cases that make the problem interesting. For example, suppose φ takes U into a polygon inscribed in the unit circle. Then the operator C_φ will be compact on H^2, even if $\|\varphi\|_\infty = 1$ (see [18], Corollary 3.2, and Section 2.4 of this paper). Additional subtlety arises from the fact that this result persists even when the corners of the inscribed polygon are rounded a bit ([18], Sec. 4). But the corners cannot be rounded too much: Schwartz ([16], page 23) observed that the mapping $\varphi(z) = (1 + z)/2$, which takes the unit disc conformally onto a subdisc touching the unit circle only at the point 1, induces on H^2 a noncompact composition operator.

This example of Schwartz was explained from a more general point of view by the author and Taylor in [18], where, for the first time, the compactness of C_φ was connected with a classical function theoretic quantity: the *angular derivative* of φ. The result is the following necessary condition for compactness.

THEOREM ([18], Theorem 2.1). *If C_φ is compact on H^2, then φ cannot have a finite angular derivative at any point of ∂U.*

Nonexistence of the angular derivative (henceforth called *the angular derivative criterion*) is not, however, sufficient for compactness on H^2. Indeed, it is well-known that inner functions induce noncompact composition operators on H^2 (see [12], Section 3.7; [13], [16]), even though some of them satisfy the angular derivative criterion ([7], [12], [21]). However the angular derivative criterion *does* characterize the compactness of composition operators on the Bergman spaces of the unit disc ([12], Theorem 3.5), and it does the same for H^2 if a mild restriction (satisfied, for example, if φ is boundedly valent) is placed on the multiplicity of φ ([12], Theorem 3.10).

In this paper, unless otherwise specified, we make no extra assumptions about φ: it need only be a holomorphic self-map of U. We do *not*, for example, require it to obey any valency restrictions, or to be continuous at any point of the boundary of U. The goal is to find a function theoretic quantity that plays for H^2 the role enjoyed in the Bergman setting by the angular derivative. Our solution involves the *Nevanlinna counting function* for φ, defined for all $w \in U \setminus \{\varphi(0)\}$ by:

$$N_\varphi(w) = \sum\{-\log|z| : z \in \varphi^{-1}\{w\}\},$$

where $\varphi^{-1}\{w\}$ denotes the sequence of φ-preimages of w, each point being repeated in the sequence according to its multiplicity.

Our main result, officially stated as Theorem 2.3, is:

$$\|C_\varphi\|_e^2 = \limsup N_\varphi(w)/(-\log|w|) \qquad (|w| \to 1-) \tag{1}$$

where $\|C_\varphi\|_e$ is the essential norm of C_φ, viewed as an operator on H^2. Since behavior near the boundary is what concerns us here, the quantity $-\log|w|$ is best imagined to be $1 - |w|$, the distance from w to the boundary. In Section 3 we will see that the expression on the right side of equation (1) dominates the supremum of a sort of "reciprocal angular derivative with multiplicity counted".

Equation (1) yields the following solution to the compactness problem:

C_φ *is compact on* $H^2 \Leftrightarrow \lim N_\varphi(w)/(-\log|w|) = 0 \qquad (|w| \to 1-)$.

Our preoccupation with H^2 to the exclusion of other Hardy spaces is explained by the fact that if C_φ is compact on H^p for some $0 < p < \infty$, then it is compact on H^p for all such p ([18], Theorem 6.1). So the solution given above for the H^2 compactness problem actually works for all H^p $(0 < p < \infty)$. For H^∞, the space of bounded holomorphic functions on U, the situation is much more elementary: C_φ *is compact on* H^∞ *if and only if* $\|\varphi\|_\infty < 1$ ([16], Theorem 2.8). Further results on the compactness problem for such "small" spaces of holomorphic functions can be found in [17].

This paper is organized as follows. In the next section we explain the motivation for our main result. Fittingly enough, this is provided by an inequality due to Littlewood ([9], [10]):

If $\varphi(0) = 0$, *then* $N_\varphi(w) \leq -\log|w|$ *for all* $w \in U \setminus \{0\}$.

Littlewood's inequality, which actually sharpens the Schwarz lemma, leads to a proof of the boundedness of C_φ that is the "right one", in that it provides the intuition behind our derivation of formula (1). We defer the proof of this formula to Sections 4 and 5, preferring to devote the remainder of Section 2, and all of Section 3 to its applications.

These applications commence with Section 2.4, where formula (1), along with Lehto's generalization [9] of Littlewood's inequality, provides an upper bound for the essential norm, expressed in terms of asymptotic behavior of the *Green function* of $\varphi(U)$. Then we use formula (1), along with the case of equality in Littlewood's inequality, to calculate the essential norm of a composition operator induced by an inner function. It turns out (perhaps not surprisingly) that such an operator is as far from compact as possible: Its essential norm coincides with its norm.

Section 3 develops the connection between counting functions and angular derivatives. The key here, as in all discussions of the angular derivative, is the Julia-Carathéodory theorem. The main result of this section is Theorem 3.3, which gives a lower bound for the essential norm of a composition operator in terms of the angular derivative of the inducing function. This result implies the necessity of the angular derivative criterion for compactness, and extends to arbitrary φ an inequality previously obtained by Carl Cowen ([3], Theorem 2.4) for functions regular at the boundary. It also yields a new proof of Oberlin and Novinger's result [14]: *Holomorphic Lipschitz functions have finite peak sets.*

Since the angular derivative criterion does not characterize compactness, there can be no upper bound on the essential norm similar to the lower one of Theorem 3.3. However, in Theorem 3.5 we obtain an interesting upper bound by restricting the multiplicity of φ for values near the boundary. This yields the fact, also noted in [12], that the angular derivative criterion characterizes compactness on H^2 for composition operators induced by boundedly valent functions.

The technical results needed to apply the Nevanlinna counting function to the study of composition operators are collected in Section 4. In particular, there is a proof of Littlewood's inequality, as well as a discussion of the case of equality. Most of these results, although known to specialists, tend to be scattered about in the literature. The idea here is to present them, along with their proofs, in one place.

The proof of formula (1) occupies Section 5.

In the sixth section we apply our techniques to *weighted Bergman spaces.* In this context it is still true that every composition operator is bounded. We show that each weighted Bergman space has its own version of the counting function which, as in the case of H^2, provides a tight estimate of the essential norm (we can no longer prove equality). There is again a lower estimate of the essential norm of C_φ in terms of the angular derivative of φ; but in contrast with the H^2 situation, the Bergman setting also provides an effective upper estimate *for arbitrary* φ. Together these estimates yield the previously mentioned equivalence, proved in [12], between Bergman compactness of C_φ and the angular derivative criterion for φ. This final section concludes with some remarks about the application of our methods to weighted Dirichlet spaces.

Throughout this paper, displayed formulas are numbered consecutively within each subsection, starting each time with (1). For example, "formula 2.4(3)" refers to formula (3) of Section 2.4. The end of a proof is marked by the symbol ///.

Before proceeding further, I would like to acknowledge the debt this work owes to Charles S. Stanton, from whose remarkable formula for integral means of holomorphic functions ([6], [19], [20]) comes the connection between composition operators and counting functions. I would also like to thank Sheldon Axler for convincing me that Carleson measures are best approached via pseudo-hyperbolic discs. Although neither Carleson measures nor pseudo-hyperbolic discs appear explicitly in this paper, they in fact lurk everywhere behind the scenes. Finally, I thank Paul Bourdon and Lech Drewnowski, for their keen criticism of a preliminary version of this manuscript.

2. The essential norm of C_φ on H^2

The Hardy space H^2 is the Hilbert space of functions f holomorphic in U for which

$$\|f\|_2^2 = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty .$$

In this paper φ will always denote a holomorphic function taking U into itself. Once more we emphasize that unless it is otherwise stated, no extra assumptions will be placed on φ.

In this section we develop in some detail the background which makes plausible our main result, Theorem 2.3. After stating this result, we devote the remainder of the section to its applications, deferring the proof to Sections 4 and 5.

A word about notation: In this paper an unadorned integral sign always denotes integration extended over the entire unit disc: $\int = \int_U$.

2.1. *The change of variable formula*. The connection between composition operators and counting functions arises directly from the following formula, a more general version of which will be proved in Section 4;

$$\|f \circ \varphi\|_2^2 = 2\int |f'|^2 N_\varphi \, d\lambda + |f(\varphi(0))|^2 \qquad (f \in H^2). \tag{1}$$

This result is also a special case of C. S. Stanton's formula for integral means ([6], [19], [20]). The special case $\varphi(z) \equiv z$ of (1) is the *Littlewood-Paley identity*:

$$(2) \qquad \|f\|_2^2 = \int |f'(z)|^2 \log(1/|z|^2)\, d\lambda(z) + |f(0)|^2 \qquad (f \in H^2),$$

which can (and should: see Section 4.5(a)) be derived more directly from a calculation with power series. The next inequality relates these two formulas, and provides the intuition behind the solution of the essential norm problem.

2.2. *Littlewood's inequality*. In [10], where he proved the boundedness of composition operators on Hardy spaces, Littlewood also proved the following inequality (see also [9]):

$$(1) \qquad N_\varphi(w) \le \log\left|\frac{1 - \overline{\varphi(0)}w}{\varphi(0) - w}\right| \qquad (w \in U \setminus \{\varphi(0)\}).$$

This is actually an improvement of the Schwarz lemma, since if $\varphi(0) = 0$, then (1) asserts that for each w in $U \setminus \{\varphi(0)\}$,

$$(2) \qquad N_\varphi(w) \le -\log|w|,$$

which, after a little rearranging, becomes:

$$\Pi\{|z|: z \in \varphi^{-1}\{w\}\} \ge |w|.$$

By contrast, the Schwarz lemma merely asserts that $|z| \ge |w|$ for each individual $z \in \varphi^{-1}\{w\}$.

To see how these results yield the boundedness of composition operators, let us temporarily continue to assume that φ fixes the origin. Then formulas (1) and (2) of the last section, along with Littlewood's inequality in the form (2) above, show that $\|f \circ \varphi\|_2 \le \|f\|_2$ for each $f \in H^2$, so that C_φ is a bounded operator on H^2 with norm ≤ 1 (in fact, since composition operators fix the constant functions, the norm is exactly 1). If φ does not fix the origin, then this argument can be applied to the composition of φ with an appropriate conformal automorphism of U. Since conformal automorphisms induce composition operators on H^2 that are surjective isomorphisms (a direct calculation), it follows easily that C_φ is still bounded, with norm depending on $\varphi(0)$.

In summary: The change of variable formula 2.1(1) shows that the boundedness of C_φ on H^2 arises from Littlewood's inequality. Since this inequality can be stated informally (at least when $\varphi(0) = 0$), as:

$$N_\varphi(w) = O(-\log|w|) \quad \text{as } |w| \to 1-,$$

it is therefore natural to conjecture that C_φ should be compact on H^2 if and only if N_φ satisfies the corresponding "little oh" condition, and that, more generally, the essential norm of C_φ should be connected with the asymptotic upper bound of $N_\varphi(w)/(-\log|w|)$.

The next result, which is the main goal of this paper, asserts that this intuition is correct, and precise.

2.3. MAIN THEOREM. *Let $\|C_\varphi\|_e$ denote the essential norm of C_φ, regarded as an operator on H^2. Then*

$$\|C_\varphi\|_e^2 = \limsup N_\varphi(w)/(-\log|w|) \qquad (|w| \to 1-).$$

In particular, C_φ is compact on H^2 if and only if

$$\lim N_\varphi(w)/(-\log|w|) = 0 \qquad (|w| \to 1-).$$

This result will be proved in Section 5, after the work of Section 4 has developed the necessary properties of counting functions. We devote the rest of the present section to immediate applications of Theorem 2.3, the first of which connects composition operators with potential theory.

2.4. *Essential norms and Green functions.* Suppose Ω is a domain contained in the unit disc, and $w_0 \in \Omega$. Let $g_\Omega(w, w_0)$ denote the Green function of Ω with singularity at w_0. Lehto's Majorization Principle ([9], [20]) asserts:

If $\varphi(U) \subset \Omega$, then $N_\varphi(w) \le g_\Omega(w, \phi(0))$ for every $w \in \Omega$.

Littlewood's Inequality 2.2(1) is the special case $\Omega = U$ of this result. Lehto's Principle and Theorem 2.3 yield:

COROLLARY. *Suppose $\varphi(U) \subset \Omega \subset U$. Then:*

$$\|C_\varphi\|_e^2 \le \limsup g_\Omega(w, \varphi(0))/(-\log|w|) \qquad (|w| \to 1-).$$

In particular, if

$$g_\Omega(w, \varphi(0)) = o(-\log|w|) \quad \text{as } |w| \to 1-,$$

then C_φ is compact on H^2.

Let us apply this last result to one of the situations mentioned in the Introduction. Suppose Ω is the interior of a polygon inscribed in the unit circle. Then

$$g_\Omega(w, \varphi(0)) = O(-\log|w|)^\gamma \qquad (|w| \to 1-)$$

for some $\gamma > 1$, so that ([18], Corollary 3.2):

COROLLARY. *C_φ is compact whenever φ takes U into a polygon inscribed in the unit circle.*

We close this section by using Theorem 2.3 to compute the essential norm of a composition operator induced by an inner function. The key here is that if φ is inner, then equality is attained in Littlewood's inequality at most points of the unit disc (see Section 4.2).

THEOREM 2.5. *Suppose φ is an inner function. Then:*

$$\|C_\varphi\|_e = \left[\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right]^{1/2}.$$

Proof. Let $a = \varphi(0)$, and define the corresponding conformal automorphism by:

$$\psi_a(w) = \frac{a-w}{1-\bar{a}w} \qquad (w \in U).$$

Then Littlewood's inequality 2.2(1) asserts that:

$$N_\varphi \le \log|\psi_a| \tag{1}$$

everywhere on $U \setminus \{a\}$. In Section 4.2 we will see that since φ is inner, there is actually *equality* in (1) on a dense subset of U (in fact on the complement of a set of capacity zero). Thus the quotient of each side of (1) with $-\log|w|$ has the same lim sup as $|w| \to 1-$; so by Theorem 2.3:

$$\|C_\varphi\|_e^2 = \overline{\lim_{|w|\to 1-}} \frac{-\log|\psi_a(w)|}{-\log|w|}. \tag{1}$$

Now a fundamental identity for conformal automorphisms is ([8], page 3):

$$1-|\psi_a(w)|^2 = \frac{(1-|a|^2)(1-|w|^2)}{|1-\bar{a}w|^2} \qquad (w \in U). \tag{2}$$

In (1) we may replace $-\log|w|$ by $1-|w|$, and $-\log|\psi_a(w)|$ by $1-|\psi_a(w)|$. Then (2) yields:

$$\|C_\varphi\|_e^2 = \overline{\lim_{|w|\to 1-}} \frac{1-|\varphi(0)|^2}{\left|1-\overline{\varphi(0)}w\right|^2}. \tag{3}$$

If $\varphi(0) = 0$ then (3) asserts that the essential norm of C_φ is 1, as desired. In any case, the right side of (3) is dominated by

$$\frac{1-|\varphi(0)|^2}{(1-|\varphi(0)|)^2} = \frac{1+|\varphi(0)|}{1-|\varphi(0)|},$$

an upper bound that, if $\varphi(0) \neq 0$, is approached arbitrarily closely as w tends to ∂U along the radius through $\varphi(0)$. ///

Nordgren ([13], Theorem 1) calculated the norm of a composition operator induced by an inner function: it coincides with the value we just found for the essential norm. This shows that such composition operators are, in some sense, as non-compact as possible.

3. Essential norms and angular derivatives

We write $\varphi^*(\zeta)$ for the nontangential limit (when it exists) of φ at the point $\zeta \in \partial U$. By Fatou's theorem this limit exists at almost every point of ∂U. We say φ has a *finite angular derivative* at a point $\zeta \in \partial U$ if there is a point $\omega \in \partial U$ such that the difference quotient $(\varphi(z) - \omega)/(z - \zeta)$ has a finite limit as z tends nontangentially to ζ. This limit, if it exists, is called the *angular derivative of* φ *at* ζ, and is denoted by $\varphi'(\zeta)$. Note that in this case $\varphi^*(\zeta) = \omega$. It is worth reiterating this important and convenient peculiarity of the above definition: *In order to have a finite angular derivative at a point of the boundary,* φ *must necessarily have an angular limit of modulus* 1 *there.*

The following classical result forges the link between composition operators and angular derivatives.

3.1. THE JULIA-CARATHÉODORY THEOREM ([2], Sec. 298, Theorem 2.1). *For* $\zeta \in \partial U$, *the following three conditions are equivalent*:

(a) φ *has finite angular derivative at* ζ.

(b) φ *has a nontangential limit of modulus* 1 *at* ζ, *and the complex derivative* φ' *has a finite nontangential limit at* ζ. *In this case the limit of* φ' *is* $\varphi'(\zeta)$.

(c) $\liminf\{(1 - |\varphi(z)|)/(1 - |z|)\colon z \to \zeta$ *unrestrictedly in* $U\} = d < \infty$. *In this case,* $\varphi'(\zeta) = \varphi^*(\zeta)\bar{\zeta} d$.

According to the Schwarz lemma, the quantity d mentioned in part (c) is never zero. Thus *the angular derivative can never be zero*. Some geometric consequences of the Julia-Carathéodory theorem are collected in the next corollary, where we use the following notation for nontangential approach regions: For $0 < \rho < 1$, let $A_\rho(\zeta)$ be the convex hull of the disc ρU and the point ζ. For $0 < r < 1$, let $A_{\rho,r}(z) = A_\rho(z) \setminus rU$.

3.2. COROLLARY. *Suppose* φ *has finite angular derivative at* $\zeta \in \partial U$. *Write* $\omega = \varphi^*(\zeta)$. *Then* φ *is conformal at* ζ *in the sense that a curve in* U *terminating at* ζ *at an angle* $\alpha < \pi/2$ *with the radius to* ζ, *is taken into a curve terminating at* ω *and making there the same angle with the corresponding radius. Moreover, for each* $0 < \sigma < \rho < 1$ *there exists* $0 < r < 1$ *such that the* φ*-image of* $A_\rho(\zeta)$ *contains* $A_{\sigma,r}(\omega)$.

Proof. By part (c) of the Julia-Carathéodory theorem,

$$\arg \varphi'(\zeta) = \arg \omega - \arg \zeta.$$

The conformality of φ follows from this and a standard argument, like the one used to prove conformality of an analytic function at interior points where the derivative does not vanish. We leave the details to the reader.

For the second part of the corollary we borrow an argument that occurs in the work of Pommerenke ([15], page 291). Observe that by part (b) of the Julia-Carathéodory theorem, there exists $0 < t < 1$ such that

$$|\varphi'(z) - \varphi'(\zeta)| < |\varphi'(\zeta)|/2 \quad \text{for all } z \in A_{\rho,t}(\zeta).$$

Thus, whenever z_1 and z_2 are points in the closure of $A_{\rho,t}(\zeta)$ and L is the line segment joining them, we have:

$$\begin{aligned} |\varphi(z_2) - \varphi(z_1) - \varphi'(\zeta)(z_2 - z_1)| &= \left| \int_L [\varphi'(z) - \varphi'(\zeta)]\, dz \right| \\ &\leq \tfrac{1}{2}|\varphi'(\zeta)||z_2 - z_1|, \end{aligned}$$

whereupon:

$$|\varphi(z_2) - \varphi(z_1)| \geq \tfrac{1}{2}|\varphi'(\zeta)||z_2 - z_1| > 0.$$

This shows that φ is univalent on the U-closure of $A = A_{\rho,t}(\zeta)$, so that $\varphi(A)$ is a simply connected open set in U bounded by the image of the three original boundary curves. By conformality, φ maps the two boundary lines of A that terminate at ζ to curves in U terminating at ω and making the same angles with the radius to ω. Thus $\varphi(A)$ contains a piece of the slightly thinner nontangential region $A_\sigma(\omega)$. ///

In view of this last result, it is not surprising that there should be some relationship between the angular derivative of φ and compactness of C_φ. For if φ has an angular derivative at ζ, then conformality requires that $\varphi(z)$ cannot be far from the unit circle when z is near ζ; so by the "intuitive principle" of the introduction, we should not expect C_φ to be compact. This is exactly what was proved in [18], Theorem 2.1: Our next result gives a quantitative version. To state it efficiently, we use the following notation. For $\omega \in \partial U$ we define

$$E(\varphi, \omega) = \{\zeta \in \partial U, \varphi^*(\zeta) = \omega\},$$

with the understanding that this set is empty if ω is not a nontangential limiting value of φ. Now define for $\omega \in \partial U$:

$$\delta(\omega) = \sum \left\{ |\varphi'(\zeta)|^{-1} : \zeta \in E(\varphi, \omega) \right\},$$

where we say that $|\varphi'(\zeta)| = \infty$ if φ does not have a finite angular derivative at

ζ, and interpret $1/\infty$ to be zero. Note that this definition is consistent with part (c) of the Julia-Carathéodory theorem.

The quantity $\delta(\omega)$ appears in the work of Cowen and Pommerenke ([3], [4]), about which we will say more in a moment. Since $E(\varphi, \omega)$ can be uncountable, it might appear that $\delta(\omega)$ could frequently be infinite if φ is sufficiently wild. The next result shows that this is not the case.

3.3. THEOREM. $\|C_\varphi\|_e^2 \geq \sup\{\delta(\omega) \colon \omega \in \partial U\}$.

In [3], Cowen obtained this result, along with an upper bound of the same type, for functions φ whose derivative extends continuously to the closed unit disc. In [4], Cowen and Pommerenke obtained sharp inequalities for $\delta(\omega)$ over the class of all holomorphic self-maps φ of the unit disc.

Proof. Fix $\omega \in \partial U$ a nontangential limiting value of φ, and suppose $\zeta_1, \zeta_2, \ldots, \zeta_n$ are points in $E(\varphi, \omega)$ at which φ has finite angular derivative. Fix $0 < \rho < 1$, and choose $0 < t < 1$ so that the angular regions $A_k = A_{\rho, t}(\zeta_k)$ are disjoint for $1 \leq k \leq n$. Corollary 3.2 insures that $\bigcap\{\varphi(A_k) \colon 1 \leq k \leq n\}$ contains an angular region A of the same type, with vertex ω.

For $w \in A \setminus \{\varphi(0)\}$ and $1 \leq k \leq n$, choose a preimage $z^{(k)}(w)$ of w that lies in A_k. Then:

$$N_\varphi(w) \geq \sum_{k=1}^{n} -\log|z^{(k)}(w)|. \tag{1}$$

For each fixed k, the Schwarz lemma insures that $z^{(k)}(w) \to \zeta_k$ through A_k as $w \to \omega$ through A. Thus, by the Julia-Carathéodory theorem:

$$\lim\left[\left(-\log|z^{(k)}(w)|\right)/\left(-\log|w|\right)\right] = |\varphi'(\zeta_k)|^{-1} \qquad (w \to \omega,\ w \in A). \tag{2}$$

Applying Theorem 2.3 along with (1) and (2) above, we obtain:

$$\begin{aligned}
\|C_\varphi\|_e^2 &\geq \limsup \sum_{k=1}^{n} \left(-\log|z^{(k)}(w)|\right)/\left(-\log|w|\right) && (w \to \omega,\ w \in A)\\
&= \sum_{k=1}^{n} \lim\left(-\log|z^{(k)}(w)|\right)/\left(-\log|w|\right) && (w \to \omega,\ w \in A)\\
&= \sum_{k=1}^{n} |\varphi'(\zeta_k)|^{-1} && \text{(by (2) above)}. \quad ///
\end{aligned}$$

A consequence of Theorem 3.3 is: *For each $\omega \in \partial U$ there can be at most countably many points in $E(\varphi, \omega)$ at which the angular derivative of φ is finite.* This also follows from the work of Cowen and Pommerenke ([4], Theorem 8.1).

More can be said about $E(\varphi, \omega)$ if φ obeys a Lipschitz condition of order 1 on U, that is, if there exists a finite constant B such that

(3) $$|\varphi(z) - \varphi(z')| \leq B|z - z'| \quad \text{for all } z, z' \in U.$$

We are going to show that in this case $E(\varphi, \omega)$ can contain at most *finitely many* points. In fact its cardinality is bounded. For a precise statement, let $\Lambda(\varphi)$ denote the smallest number B for which (3) is valid.

3.4. THEOREM. *Suppose φ obeys a Lipschitz condition of order 1 on U. Then for each $\omega \in \partial U$ the set $E(\varphi, \omega)$ has at most $\Lambda(\varphi)\|C_\varphi\|_e^2$ points.*

Proof. Since φ obeys a Lipschitz condition, it extends continuously to the closed unit disc: Denote this extension again by φ. Suppose $\zeta \in \partial U$ is a point for which $\varphi(\zeta) \in \partial U$. Then for $0 \leq r < 1$:

$$\left(1 - |\varphi(r\zeta)|\right)/(1 - r) \leq |\varphi(\zeta) - \varphi(r\zeta)|/|\zeta - r\zeta| \leq \Lambda(\varphi).$$

Thus the Julia-Carathéodory theorem insures that φ has a finite angular derivative at ζ, of magnitude $\leq \Lambda(\varphi)$.

Now suppose $\zeta_1, \zeta_2, \ldots, \zeta_n$ are distinct points of $E(\varphi, \omega)$. Then by Theorem 3.3 and the bound just obtained on the angular derivative of φ:

$$n/\Lambda(\varphi) \leq \delta(\omega) \leq \|C_\varphi\|_e^2,$$

as desired. ///

Using different methods, Cowen ([3], Corollary, page 84) proved this result for the slightly more restrictive class of functions φ whose derivative extends continuously to the closed unit disc. By contrast, the holomorphic Lipschitz functions are precisely those for which the derivative is *bounded* on U ([5], Theorem 5.1, page 74).

Theorem 3.4, is, in fact, a result on peak sets originally due to Novinger and Oberlin [14]: *The peak set of a holomorphic Lipschitz function has at most finitely many points.* Our contribution has been to provide a new proof.

We close this section by deriving an upper bound for the essential norm in terms of the angular derivative. As was pointed out in the introduction, some restriction on the inducing function is required. To state it, let $n_\varphi(w)$ denote the number of points in $\varphi^{-1}\{w\}$, with multiplicity counted.

3.5. THEOREM. *Suppose that for some $0 < R < 1$:*

(1) $$\sup\left\{n_\varphi(w) \colon R < |w| < 1\right\} = M < \infty.$$

Then:

$$\|C_\varphi\|_e^2 \leq M \sup\left\{|\varphi'(\zeta)|^{-1} \colon \zeta \in \partial U\right\}.$$

Proof. By Theorem 2.3 there exists a sequence (w_n) in U such that $R < |w_n| \to 1-$, and

$$\|C_\varphi\|_e^2 = \lim N_\varphi(w_n)/(-\log|w_n|) \qquad (n \to \infty). \tag{2}$$

By (1), the pre-image sequence $\varphi^{-1}\{w_n\}$ has at most M points; so the sum in the definition of $N_\varphi(w_n)$ contains at most M terms. Let $z^{(n)}$ denote the point in $\varphi^{-1}\{w_n\}$ that has the smallest modulus. Then for all n:

$$N_\varphi(w_n) \le M(-\log|z^{(n)}|). \tag{3}$$

The Schwarz lemma insures that $|z^{(n)}| \to 1$ as $n \to \infty$; by selecting an appropriate subsequence, if necessary, we may assume that $(z^{(n)})$ converges to some point $\zeta \in \partial U$. By (2) and (3) along with the Julia-Carathéodory theorem:

$$\begin{aligned}\|C_\varphi\|_e^2 &\le M \lim(-\log|z^{(n)}|)/(-\log|w_n|) \qquad (n \to \infty)\\ &\le M \limsup(1-|z|)/(1-|\varphi(z)|) \qquad (z \to \zeta)\\ &= M|\varphi'(\zeta)|^{-1} \qquad \text{(Julia-Carathéodory Theorem)}.\end{aligned}$$

The proof is now complete. ///

3.6. Corollary. *If φ satisfies both the angular derivative criterion and the multiplicity restriction* (1) *of Theorem* 3.5, *then C_φ is compact on H^2.*

A result that implies this last one was proved in [12] (Theorem 3.10). An interesting problem is to try to find concrete multiplicity restrictions weaker than the one above under which the angular derivative condition implies H^2-compactness. One such result will be given at the end of Section 6.

4. The Nevanlinna counting function

In this section we develop those properties of the Nevanlinna counting function needed for the study of composition operators on H^2. In order of appearance, these are: Littlewood's inequality, the change of variable formula, and the subharmonic mean value property.

4.1. *Partial counting functions.* For $w \in \mathbf{C} \setminus \{\varphi(0)\}$, let $\{z_j(w): j \ge 1\}$ denote the points of the preimage $\varphi^{-1}\{w\}$, arranged in order of increasing moduli, with each point repeated according to its multiplicity. For $0 \le r < 1$, let $n(r, w) = n_\varphi(r, w)$ denote the number of these points in the disc rU, and define the *partial counting functions* for φ by:

$$N_\varphi(r, w) = \sum_{j=1}^{n(r,w)} \log(r/|z_j|).$$

With this notation the original counting function defined in Section 1 can be regarded as $N_\varphi(1, w)$:

$$N_\varphi(w) = N_\varphi(1, w) = \sum_j \log\big(1/|z_j(w)|\big).$$

Our understanding will be that for $0 \le r \le 1$, $N_\varphi(r, w) = 0$ whenever $w \notin \varphi(rU)$, so that our counting functions can be regarded as defined on the entire complex plane. Note that for each fixed complex number w, the partial counting function $N_\varphi(r, w)$ increases with r. It follows easily from monotone convergence that $\lim N_\varphi(r, w) = N_\varphi(w)$ $(r \to 1-)$.

4.2. *Littlewood's inequality* ([9], [10]). *For each w in $U \setminus \{\varphi(0)\}$:*

$$N_\varphi(w) \le \log\left|\frac{1 - \overline{\varphi(0)}w}{\varphi(0) - w}\right|. \tag{1}$$

Regarding the case of equality in (1), *the following are equivalent*:

(a) *There is equality for some w.*

(b) *There is equality for all w outside a subset of U having logarithmic capacity zero.*

(c) *φ is an inner function.*

Proof. Jensen's formula ([8], page 54) asserts that for each function f holomorphic on U, with $f(0) \neq 0$, and for each $0 \le r < 1$:

$$N_\varphi(r, 0) = \int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta/2\pi - \log|f(0)|. \tag{2}$$

For φ, as always, taking U into itself, write

$$\varphi_w(z) = \frac{w - \varphi(z)}{1 - \overline{w}\varphi(z)} \qquad (z \in U),$$

so that φ_w is holomorphic on U, takes U into itself, and has $\varphi^{-1}\{w\}$ as its zero set (multiplicities included). Upon applying (2) with $f = \varphi_w$ where $w \neq \varphi(0)$, we obtain:

$$N_\varphi(r, w) = N_{\varphi_w}(r, 0) = \frac{1}{2\pi}\int_0^{2\pi} \log|\varphi_w(re^{i\theta})|\, d\theta - \log|\varphi_w(0)|. \tag{3}$$

Since $|\varphi_w| \le 1$ on U, the integral on the right side of (3) is negative; so for all w in $U \setminus \{\varphi(0)\}$,

$$N_\varphi(r, w) \le -\log|\varphi_w(0)|.$$

Littlewood's inequality (1) follows immediately from this as r tends to 1.

The case of equality in (1): By (3) equality holds if and only if

$$\lim \int_0^{2\pi} \log |\varphi_w(re^{i\theta})|\, d\theta = 0 \qquad (r \to 1-). \tag{4}$$

Since $|\varphi_w| \le 1$ on U, this condition is equivalent to ([8], Theorem 2.4, page 56):

$$\varphi_w \text{ is a Blaschke product.} \tag{4'}$$

Now φ_w is inner if and only if φ is; and Frostman's theorem (see [8], Theorem 6.4, page 79, for example) asserts that if φ is inner then φ_w is a Blaschke product for every $w \in U$, with the possible exception of a set of logarithmic capacity zero. These remarks yield statements (a), (b), and (c) above. ///

The next result establishes the connection between counting functions and composition operators. Recall that λ is planar Lebesgue measure, normalized so that the area of the unit disc is 1. In what follows, λ_1 denotes the probability measure defined on U by: $d\lambda_1(w) = \log(1/|w|^2)\, d\lambda(w)$. Recall that $\int = \int_U$.

4.3. *Change of variable formula. If g is a positive measurable function on U, then*:

$$\int (g \circ \varphi)|\varphi'|^2\, d\lambda_1 = 2\int g N_\varphi\, d\lambda.$$

Proof. Since φ is a local homeomorphism on the open set U' formed by deleting from U the zeros of the derivative φ', there exists a countable collection $\{R_j\}$ of pairwise disjoint, semi-closed polar rectangles whose union is U', and such that φ is one-to-one on each R_j. Let ψ_j denote the inverse of the restriction of φ to R_j, so that ψ_j is a one-to-one map taking $\varphi(R_j)$ back onto R_j. By the usual change of variable formula applied on R_j, with $z = \psi_j(w)$:

$$\int_{R_j} g(\varphi(z))|\varphi'(z)|^2\, d\lambda_1(z) = \int_{\varphi(R_j)} g(w) \log\left(1/|\psi_j(w)|^2\right) d\lambda(w).$$

Thus, if χ_j denotes the characteristic function of the set $\varphi(R_j)$:

$$\int (g \circ \varphi)|\varphi'|^2\, d\lambda_1 = 2\int g(w)\left\{\sum_j \chi_j(w) \log\left(1/|\psi_j(w)|\right)\right\} d\lambda(w).$$

This is the desired formula, since the term in curly braces on the right side of the equation above is $N_\varphi(w)$. ///

As mentioned in the introduction, we require only the following special case of the change of variable formula, which also follows from C. S. Stanton's formula for integral means ([6], [19], [20]).

4.4. COROLLARY. *For each f holomorphic on U;*

$$\|f \circ \varphi\|_2^2 = 2\int |f'|^2 N_\varphi \, d\lambda + |f(\varphi(0))|^2.$$

Proof. The Littlewood-Paley identity 2.1(2) applied to $f \circ \varphi$ yields:

$$\begin{aligned}\|f \circ \varphi\|_2^2 &= \int |(f \circ \varphi)'|^2 \, d\lambda_1 + |f(\varphi(0))|^2 \\ &= \int |f' \circ \varphi|^2 |\varphi'|^2 \, d\lambda_1 + |f(\varphi(0))|^2 \qquad \text{(chain rule)}.\end{aligned}$$

An application of Theorem 4.3, with $g = |f'|^2$, completes the proof. ///

4.5. *Remarks.* (a) As we pointed out in Section 2, the Littlewood-Paley identity is the special case $\varphi(z) \equiv z$ of Corollary 4.4. However, since it plays an essential role in the derivation of Corollary 4.4 from Theorem 4.3, a separate proof is desirable. This can easily be furnished either by a power series calculation or by the use of Green's Theorem (see [8], Lemma 3.1, page 236, for example).

(b) The special case $f(z) \equiv z$ of Corollary 4.4 yields the formula:

$$\|\varphi\|_2^2 = 2\int N_\varphi \, d\lambda + |\varphi(0)|^2,$$

from which follows an estimate that will be needed later on:

$$\int N_\varphi \, d\lambda \leq \left(1 - |\varphi(0)|^2\right)/2 \leq \tfrac{1}{2}.$$

(c) A result similar to Theorem 4.3 occurs in [1], where it is used to examine the boundedness of composition operators on certain Besov spaces. We will use a variation of Theorem 4.3 in Section 6 when we study composition operators on Bergman spaces.

(d) With the notation: $d\mu(z) = |\varphi'(z)|^2 \, d\lambda_1(z)$, the change of variable formula 4.3 can be restated:

$$d\mu\varphi^{-1}(w) = 2N_\varphi(w) \, d\lambda(w).$$

The measure $\mu\varphi^{-1}$ plays a crucial role in [12], as well as in the application to composition operators of Luecking's trace ideal criterion [11].

4.6. *The subharmonic mean value property* ([6], *Section* 2). *Suppose g is holomorphic on a plane region* Ω, *and* Δ *is an open disc in* $\Omega \setminus g^{-1}\{\varphi(0)\}$ *with center a. Then*:

$$N_\varphi(g(a)) \leq \frac{1}{\lambda(\Delta)} \int_\Delta N_\varphi(g(w)) \, d\lambda(w).$$

Proof. For $0 \leq r < 1$, write $\varphi_r(z) = \varphi(rz)$ for $z \in U$. Set $d\sigma = d\theta/2\pi$, and let μ be the Borel probability measure defined on $\mathbf{C}$ (but supported only on $\varphi_r(\partial U)$) by $\mu = \sigma\varphi_r^{-1}$. Then for each fixed complex number $w \neq \varphi(0)$, Jensen's formula 4.2(2), with $f = \varphi - w$, can be rewritten:

$$(1) \qquad N_\varphi(r, w) + \log|\varphi(0) - w| = \int \log|\zeta - w|\, d\mu(\zeta).$$

The left side of (1), being the logarithmic potential of a compactly supported probability measure, is therefore subharmonic in the plane; hence $N_\varphi(r, w)$ is a subharmonic function of w in $\mathbf{C} \setminus \{\varphi(0)\}$. Therefore $N_\varphi(r, g(w))$ is subharmonic on $\Omega \setminus g^{-1}\{\varphi(0)\}$, and as r increases to 1, these functions increase pointwise on that set to $N_\varphi(g(w))$. Thus the Monotone Convergence Theorem insures that $N_\varphi(g(w))$ inherits from $N_\varphi(r, g(w))$ the desired sub-mean value property. ///

Remark. It follows from Jensen's formula that each function $N_\varphi(r, g(w))$ is continuous in w; hence $N_\varphi(g(w))$ is lower semicontinuous on Ω. However in order to be subharmonic, it must be upper semicontinuous, hence continuous. This need not happen at every point of Ω (see [6], Section 2).

5. Proof of the Main Theorem

We can now prove Theorem 2.3: the formula for the essential norm of a composition operator on H^2. This involves separate arguments for the upper and lower estimates. For the upper one we use the following general formula for the essential norm of a linear operator on Hilbert space.

5.1. PROPOSITION. *Suppose T is a bounded linear operator on a Hilbert space H. Let $\{K_n\}$ be a sequence of compact self-adjoint operators on H, and write $R_n = I - K_n$. Suppose $\|R_n\| = 1$ for each n, and $\|R_n x\| \to 0$ for each $x \in H$. Then $\|T\|_e = \lim_n \|TR_n\|$.*

Proof. Suppose K is a compact operator on H, and n is temporarily fixed. Since $\|R_n\| = 1$,

$$(1) \qquad \|T - K\| = \|T - K\|\,\|R_n\| \geq \|(T - K)R_n\| \geq \|TR_n\| - \|KR_n\|.$$

Now R_n is self-adjoint, so that

$$(2) \qquad \|KR_n\| = \|(KR_n)^*\| = \|R_n K^*\|.$$

The sequence $\{R_n\}$ tends pointwise to zero, and is uniformly bounded in the operator norm. It is therefore equicontinuous hence convergent uniformly to zero on each relatively compact subset of H. In particular, since K^* is compact, this happens on K^*(unit ball of H): that is, $\|R_nK^*\| \to 0$. Thus by (1) and (2) above, $\|T - K\| \geq \overline{\lim}\|TR_n\|$. Since this is true for every compact operator K on H, we have: $\|T\|_e \geq \overline{\lim}_n\|TR_n\|$.

For the inequality in the other direction, we have for each n: $R_n + K_n = I$, with K_n compact. Since TK_n is compact, this yields:

$$\|T\|_e = \|TR_n + TK_n\|_e = \|TR_n\|_e \leq \|TR_n\|.$$

Since n is arbitrary, $\|T\|_e \leq \underline{\lim}_n\|TR_n\|$. ///

After the proof of Theorem 2.5 we noted that if φ is inner, then the norm of the induced composition operator coincides with its essential norm. It is an amusing exercise to use Proposition 5.1 to derive this fact without making any explicit computations. Theorem 2.5 can then be viewed as a consequence of Proposition 5.1 and Nordgren's original norm computation.

We also require the following elementary estimates on H^2 functions.

5.2. Lemma. *Suppose $f \in H^2$ has a zero of order at least n at the origin. Then for each $z \in U$*:

(a) $|f(z)| \leq |z|^n(1 - |z|^2)^{-1/2}\|f\|_2$, *and*

(b) $|f'(z)| \leq \sqrt{2}\,n|z|^{n-1}(1 - |z|^2)^{-3/2}\|f\|_2$.

Proof. We are assuming the series representation

$$f(z) = \sum_{k=n}^{\infty} a_k z^k,$$

where $\sum|a_k|^2 = \|f\|_2^2 < \infty$. Inequality (a) follows immediately from the Cauchy-Schwarz inequality. The same reasoning, applied to the series representation of the derivative yields:

$$|f'(z)| \leq \left\{\sum_{k=n}^{\infty} k^2|z|^{2(k-1)}\right\}^{1/2}\|f\|_2$$

$$= |z|^{n-1}\left\{\sum_{k=0}^{\infty} (n+k)^2|z|^{2k}\right\}^{1/2}\|f\|_2,$$

from which the desired inequality follows after some simple estimates. ///

5.3. *The upper estimate.* The goal of this section is to show that

$$\text{(1)} \qquad \|C_\varphi\|_e^2 \leq \overline{\lim}\, N_\varphi(w)/(-\log|w|) \qquad (|w| \to 1-).$$

This will be accomplished by applying Proposition 5.1 with K_n the operator that takes f to the nth partial sum of its Taylor series:

$$K_nf(z) = \sum_{k=0}^{n} a_k z^k, \qquad \text{where} \qquad f(z) = \sum_{k=0}^{\infty} a_k z^k \in H^2.$$

Since K_n is the orthogonal projection of H^2 onto the closed subspace spanned by the monomials $1, z, \ldots, z^n$, it is self-adjoint and compact. Since $R_n = I - K_n$ is the complementary projection, its norm is 1. Thus the hypotheses of Proposition 5.1 are fulfilled, so that

$$\|C_\varphi\|_e = \lim \|C_\varphi R_n\| \qquad (n \to \infty). \tag{2}$$

To estimate the right side of (2), fix for the moment a function f in the unit ball of H^2, and a positive integer n. Then by Corollary 4.4:

$$\|C_\varphi R_n f\|_2^2 = 2\int |(R_n f)'|^2 N_\varphi \, d\lambda + |R_n f(\varphi(0))|^2, \tag{3}$$

Since $\|f\|_2 \le 1$, the same is true of $R_n f$. Since $R_n f$ has a zero of order n at the origin, it can play the role of f in Lemma 5.2. The results are:

$$|R_n f(\varphi(0))|^2 \le |\varphi(0)|^{2n} / \left(1 - |\varphi(0)|^2\right), \quad \text{and} \tag{4}$$

$$|(R_n f)'(z)|^2 \le 2n^2 |z|^{2(n-1)} / (1 - |z|^2)^3 \qquad (z \in U). \tag{5}$$

Now fix $0 < r < 1$, and split the integral on the right side of (3) into two parts: one extended over the disc rU, and the other over its complement. Use estimate (4) on the last term of (3), and (5) on the integral over rU. Then take the supremum of both sides of the resulting inequality over all functions f in the unit ball of H^2. Denoting this supremum for the moment by "sup", we obtain:

$$\|C_\varphi R_n\|^2 \le 2 \sup \int_{U \setminus rU} |(R_n f)'|^2 N_\varphi \, d\lambda + \frac{4n^2 r^{2(n-1)}}{(1 - r^2)^3} \int_{rU} N_\varphi \, d\lambda + \frac{|\varphi(0)|^{2n}}{1 - |\varphi(0)|^2}. \tag{6}$$

As f traverses the unit ball of H^2, its truncation $R_n f$ runs through a subset of that ball; hence the right side of (6) can only be made larger by replacing $R_n f$ by f in the term involving the supremum. This observation, along with the fact, noted in Section 4.5(b), that $\int N_\varphi \, d\lambda \le \frac{1}{2}$, yields:

$$\|C_\varphi R_n\|^2 \le 2 \sup \int_{U \setminus rU} |f'|^2 N_\varphi \, d\lambda + \frac{2n^2 r^{2(n-1)}}{(1 - r^2)^3} + \frac{|\varphi(0)|^{2n}}{1 - |\varphi(0)|^2}.$$

Now let n tend to infinity (the radius r is still held fixed: $0 < r < 1$). Upon recalling (2), introducing the notation B for the unit ball of H^2, and writing

$$h(w) = N_\varphi(w) / (-\log|w|),$$

we have:

$$
\begin{aligned}
\|C_\varphi\|_e^2 &\le 2\sup_B \int_{U\setminus rU} |f'|^2 N_\varphi \, d\lambda \\
&= \sup_B \int_{U\setminus rU} |f'|^2 h \, d\lambda_1 \\
&\le \sup\{h(w)\colon r \le |w| < 1\} \sup_B \int_U |f'|^2 \, d\lambda_1 \\
&\le \sup\{h(w)\colon r \le |w| < 1\},
\end{aligned}
$$

where the last line follows from the Littlewood-Paley identity. Inequality (1) follows when r tends to 1. ///

5.4. *The lower estimate.* We finish the proof of Theorem 2.3 by showing that:

$$
(1) \qquad \|C_\varphi\|_e^2 \ge \overline{\lim}\, N_\varphi(w)/(-\log|w|) \quad (|w| \to 1-).
$$

To accomplish this we apply the operator C_φ to the "normalized reproducing kernels", defined for $a \in U$ by

$$
f_a(z) = \frac{(1-|a|^2)^{1/2}}{1-\bar{a}z} \qquad (z \in U).
$$

A straightforward calculation with power series shows that $\|f_a\|_2 = 1$ for each $a \in U$.

For the moment fix a compact operator K on H^2. Since the family $\{f_a\colon a \in U\}$ is bounded in H^2, and since $f_a \to 0$ uniformly on compact subsets of U as $|a| \to 1-$, we also have $\|Kf_a\|_2 \to 0$. Thus, letting "$\overline{\lim}$" denote the upper limit as $|a| \to 1-$:

$$
\begin{aligned}
\|C_\varphi - K\| &\ge \overline{\lim}\, \|(C_\varphi - K)f_a\|_2 \\
&\ge \overline{\lim}(\|C_\varphi f_a\|_2 - \|Kf_a\|_2) \\
&= \overline{\lim}\, \|C_\varphi f_a\|_2.
\end{aligned}
$$

Upon taking the infimum of both sides of this inequality over all compact operators K on H^2, we obtain the lower estimate:

$$
\begin{aligned}
(2) \qquad \|C_\varphi\|_e^2 &\ge \overline{\lim}\, \|C_\varphi f_a\|_2^2 \\
&= 2\,\overline{\lim} \int |f_a'|^2 N_\varphi \, d\lambda,
\end{aligned}
$$

where the last line follows from Corollary 4.4 and the fact that $f_a(\varphi(0)) \to 0$ as $|a| \to 1-$.

The integral on the right side of (2) is calculated for fixed $a \in U$ by use of the change of variable $w = \psi_a(z)$, where ψ_a is the conformal automorphism defined in the proof of Theorem 2.5. Here are the properties required of this automorphism.

(3) $$\psi_a(0) = a,$$

(4) $$\psi_a'(z) = (1 - |a|^2)/|1 - \bar{a}z|^2,$$

(5) $$\psi_a(\psi_a(z)) = z, \quad \text{i.e. } w = \psi_a(z) \Leftrightarrow z = \psi_a(w).$$

The last line yields the change of variable formula:

(6) $$d\lambda(z) = |\psi_a'(w)|^2 \, d\lambda(w) \quad \text{if } w = \psi_a(z).$$

The calculation of the integral can now proceed. It will be convenient to use the notation $c(a) = |a|^2/(1 - |a|^2)$:

$$\begin{aligned} \int |f_a'|^2 N_\varphi \, d\lambda &= |a|^2(1 - |a|^2) \int |1 - \bar{a}w|^{-4} N_\varphi(w) \, d\lambda(w) \\ &= c(a) \int |\psi_a'(w)|^2 N_\varphi(w) \, d\lambda(w) & \text{(by (4))} \\ &= c(a) \int N_\varphi(\psi_a(z)) \, d\lambda(z) & \text{(by (6))}. \end{aligned}$$

By Section 4.6, the composition $N_\varphi \circ \psi_a$ obeys the sub-mean value inequality on any disc that does not contain $\psi_a^{-1}(\varphi(0)) = \psi_a(\varphi(0))$. Fix $0 < r < 1$. Since $\psi_a(\varphi(0))$ tends to the boundary of U with a, it will lie outside the disc rU for all points a of modulus sufficiently close to 1. Thus for all such a, the last calculation yields:

$$\begin{aligned} \int |f_a'|^2 N_\varphi \, d\lambda &\geq c(a) \int_{rU} N_\varphi(\psi_a(z)) \, d\lambda(z) \\ &\geq c(a) N_\varphi(\psi_a(0)) \lambda(rU), \end{aligned}$$

where the last line follows from the sub-mean value inequality of Theorem 4.6. Since $\psi_a(0) = a$ and $\lambda(rU) = r^2$, the last estimate, along with (2) yields

(3) $$\|C_\varphi f_a\|_2^2 \geq 2c(a) r^2 N_\varphi(a) + |f_a(\varphi(0))|^2$$

for all points a with modulus sufficiently close to 1. Now let $|a| \to 1-$ in (3), and use (1); recalling all the while that r is fixed. The result is:

(4) $$\|C_\varphi\|_e^2 \geq 2r^2 \overline{\lim}\, c(a) N_\varphi(a) = r^2 \overline{\lim}\, 2N_\varphi(a)/(1 - |a|^2).$$

Since (4) holds for all $0 < r < 1$, it also holds for $r = 1$. The lower estimate (1) now follows from (4) (with $r = 1$) and the fact that the ratio of $2(-\log|a|)$ to $1 - |a|^2$ tends to 1 as $|a| \to 1-$. ///

The proof of our main result, Theorem 2.3, is now complete.

6. Weighted Bergman spaces

For $\alpha > -1$ let λ_α denote the probability measure defined on U by:

$$d\lambda_\alpha(z) = (-2\log|z|)^\alpha \, d\lambda(z)/\Gamma(\alpha + 1),$$

where Γ denotes the gamma function. We have already encountered the measure λ_1. In this section we develop results analogous to those of Section 2 for the *weighted Bergman spaces* A^2_α consisting of all functions f holomorphic on U for which

$$\|f\|_\alpha^2 = \int |f|^2 \, d\lambda_\alpha < \infty. \tag{1}$$

In the norm defined above, A^2_α is a Hilbert space. Since

$$\int |z|^{2n} \, d\lambda_\alpha = (n+1)^{-(\alpha+1)},$$

this norm can be computed from the power series coefficients of f:

$$\|f\|_\alpha^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2/(n+1)^\alpha \tag{2}$$

where $f(z) = \Sigma \hat{f}(n) z^n$ can be taken to be any function holomorphic on U, provided we understand that the integral in (1) is finite if and only if the same is true of the sum in (2).

In defining the space A^2_α, the measure λ_α is frequently replaced by $(1 - |z|^2)^\alpha \, d\lambda(z)$ (cf. [12]). Since the densities of these two measures are, up to multiplicative constants, equivalent as $|z| \to 1-$, and since the density for $d\lambda_\alpha$ has only an integrable singularity at the origin, the same space results, and the norms are equivalent.

In this section we show that each weighted Bergman space has its own version of the counting function. As for H^2, this counting function determines the essential norm of any composition operator.

The main point of difference between the results of this section and those for H^2 is that here the angular derivative furnishes not only a lower bound on the essential norm of C_φ, but also an upper bound that is valid for all φ. Together these bounds yield, as a special case, one of the main results of [12]: *C_φ is compact on A^2_α if and only if φ satisfies the angular derivative criterion.*

At the end of this section we comment briefly on applications of our methods to weighted Dirichlet spaces. This leads to a multiplicity restriction more general than that of Theorem 3.5 under which the angular derivative criterion implies compactness on H^2.

6.1. *A Littlewood-Paley formula.* Formula (2) of the last section shows that for f holomorphic in U:

$$\|f\|_\alpha^2 = |\hat{f}(0)|^2 + \|f'\|_{\alpha+2}^2.$$

In particular, $f \in A_\alpha^2$ if and only if $f' \in A_{\alpha+2}^2$. This formula is the Bergman space version of the Littlewood-Paley identity 2.1(2) employed in the previous sections. It should be noted that in the results above, and all the others to follow, the H^2 results show up as the limiting case $\alpha \to -1$.

6.2. *Generalized counting functions.* For $w \in \mathbf{C} \setminus \{\varphi(0)\}$, $0 \le r < 1$, and $\gamma \ge 0$, we write:

$$N_{\varphi,\gamma}(w) = \sum_{j=1}^{n(r,w)} \log\left(r/|z_j(w)|\right)^\gamma, \quad \text{and}$$

$$N_{\varphi,\gamma}(w) = N_{\varphi,\gamma}(1, w) = \sum_j \left[\log\left(1/|z_j(w)|\right)\right]^\gamma$$

where, as before, $\{z_j(w)\}$ denotes the multiplicity sequence of φ-preimages of w. To save notation, we write N_γ for $N_{\varphi,\gamma}$. Observe that $N_0(r, w)$ is the *multiplicity function* $n(r, w)$ of φ, and N_1 is the classical Nevanlinna counting function that figured so prominently in our work on H^2. We have the following extension of Littlewood's inequality (sections 2.2 and 4.2).

6.3. Proposition. *For all* $\gamma \ge 1$,

$$N_\gamma(w) \le \left(\log\left|\frac{1 - \overline{\varphi(0)}w}{\varphi(0) - w}\right|\right)^\gamma \qquad (w \in U \setminus \{\varphi(0)\}).$$

Proof. The case $\gamma = 1$ is the original Littlewood inequality. If $\gamma > 1$, then for any positive numbers (t_j) we have $\Sigma t_j^\gamma \le (\Sigma t_j)^\gamma$. Thus $N_\gamma(w) \le N_1(w)^\gamma$ for every $w \in U$; so the desired inequality follows from the case $\gamma = 1$. ///

The importance of N_γ stems from the following two results, which generalize the change of variable formula of section 4.3, and its corollary. We omit the proofs, which are almost identical with the original ones, the only difference being that version 6.1 of the Littlewood-Paley identity is used to derive Corollary 6.5 from Theorem 6.4.

6.4. CHANGE OF VARIABLE FORMULA. *If g is a positive measurable function on U, then:*

$$\int (g \circ \varphi)|\varphi'|^2 \, d\lambda_\gamma = c(\gamma) \int g N_\gamma \, d\lambda,$$

where $c(\gamma) = 2^\gamma/\Gamma(\gamma + 1)$.

6.5. COROLLARY. *If* $f \in A^2_\alpha$, *then*

$$\|f \circ \varphi\|^2_\alpha = c(\alpha + 2) \int |f'|^2 N_{\alpha+2} \, d\lambda + |f(\varphi(0))|^2.$$

The last result shows that for $\alpha > -1$ it is the counting function $N_{\alpha+2}$ that controls of the behavior of C_φ on A^2_α.

From sections 6.3 and 6.5 we obtain another proof of the well-known fact that composition operators are bounded on A^2_α. More precisely, just as for H^2: *If* $\varphi(0) = 0$, *then* C_φ *has norm* 1 *as an operator on* A^2_α. We will be able to obtain analogues of the other results for H^2 once we prove that $N_{\alpha+2}$ has the subharmonic mean value property. The key to this is the following representation formula.

6.6. PROPOSITION. *For* $\gamma > 1$, *if* $w \neq \varphi(0)$, *then*

$$N_\gamma(r, w) = \gamma(\gamma - 1) \int_0^r N_1(t, w)(\log(r/t))^{\gamma-2} \, dt.$$

Proof. Integrate by parts twice. In each case the "integrated" terms are zero because the condition $w \neq \varphi(0)$ guarantees that $n(t, w)$, and therefore $N_1(t, w)$, both vanish for all t sufficiently close to zero. The first integration by parts yields:

$$\begin{aligned} N_\gamma(r, w) &= \int_0^r (\log(r/t))^\gamma \, dn(t) \\ &= \gamma \int_0^r t^{-1} n(t, w)(\log(r/t))^{\gamma-1} \, dt. \end{aligned}$$

For the second one we use the formula above for $\gamma = 1$:

$$N_1(r, w) = \int_0^r t^{-1} n(t, w) \, dt.$$

This yields:

$$\int_0^r t^{-1} n(t, w)(\log(r/t))^{\gamma-1} \, dt = (\gamma - 1) \int_0^r N_1(t, w)(\log(r/t))^{\gamma-2} \, dt,$$

and the result follows. ///

6.7. COROLLARY. *If $\gamma > 1$ then the subharmonic mean value property of section 4.6 continues to hold with N_γ in place of N_1.*

Proof. By Proposition 6.6, $N_\gamma(r, w)$ inherits the continuity and subharmonicity of $N_1(r, w)$ on $\mathbf{C} \setminus \{\varphi(0)\}$ for each $0 \leq r < 1$. As in section 4.6, the result follows from this and the monotone convergence theorem. ///

We can now estimate the essential norm of a composition operator on a weighted Bergman space. In what follows it will be convenient to denote the essential norm of C_φ on A^2_α by $\|C_\varphi\|_{e,\alpha}$, and to write:

$$\sigma_\gamma(\varphi) = \limsup\{N_{\gamma,\varphi}(w)/(-\log|w|)^\gamma\} \qquad (|w| \to 1-).$$

Here is our analogue of Theorem 2.3:

6.8. THEOREM. *For each $\alpha > -1$ there exists a constant $m > 0$, which depends only on α, such that*

$$m\sigma_{\alpha+2}(\varphi) \leq \|C_\varphi\|^2_{e,\alpha} \leq \sigma_{\alpha+2}(\varphi).$$

Proof. The argument is modeled after the proof of Theorem 2.3, so we need only outline the high points. As before, the upper estimate uses Proposition 5.1, with K_n again taken to be the nth partial sum operator, and $R_n = I - K_n$. With B denoting the unit ball of A^2_α, the result is:

$$(1) \qquad \|C_\varphi\|^2_{e,\alpha} = \lim_n \sup_B \|C_\varphi R_n f\|$$
$$= \lim_n \sup_B c(\alpha+2) \int |(R_n f)'|^2 N_{\alpha+2}\, d\lambda,$$

where the second line arises from Corollary 6.5, and from an estimate similar to 5.2(a) which shows that as n tends to infinity, $R_n f(\varphi(0))$ converges to zero uniformly over B. As before, fix $0 < r < 1$ and split the integral on the right side of (1) into two pieces: one over rU, and the other over the complement. By an estimate similar to 5.2(b), as $n \to \infty$ the integral over rU tends to zero uniformly over B; so by (1) above:

$$(2) \qquad \|C_\varphi\|^2_{e,\alpha} = \lim_n \sup\left\{c(\alpha+2)\int_{U\setminus rU} |(R_n f)'|^2 N_{\alpha+2}\, d\lambda : f \in B\right\}$$
$$\leq \sup\left\{c(\alpha+2)\int_{U\setminus rU} |f'|^2 N_{\alpha+2}\, d\lambda : f \in B\right\},$$

where, as in Section 5, the last line follows from the fact that R_n is an operator of norm 1. The upper estimate now follows as in the proof of Theorem 5.3: by rewriting the integral on the right side of (2) in terms of $\lambda_{\alpha+2}$, replacing the quotient $N_{\alpha+2}(w)/(-\log|w|)^{\alpha+2}$ by its supremum over $U \setminus rU$, using the Littlewood-Paley formula of section 6.1, and finally letting r tend to 1.

The lower estimate proceeds along the lines of section 5.4, except that the result is not as precise. For $a \in U$, set

$$F_a(z) = [f_a(z)]^{\alpha+2} \qquad (z \in U),$$

where f_a is the normalized reproducing kernel for H^2 defined in Section 5.4. A calculation with power series based on formula (2) at the beginning of this section shows that the functions F_a form a bounded family in A^2_α (their norms are not necessarily 1 now, so we already face an inaccuracy that was not present in the H^2 situation). By the reasoning employed in section 5.4:

$$(3) \qquad \|C_\varphi\|_e \geq m \limsup \|C_\varphi F_a\|_{2,\alpha} \qquad (|a| \to 1-),$$

where $1/m = \liminf \|F_a\|_{2,\alpha}$ $(|a| \to 1-)$. Using Corollary 6.5 to calculate the norms on the right side of (3), we have for each $a \in U$:

$$(4) \quad \|C_\varphi F_a\|^2_{2,\alpha} = c(\alpha+2)\int |F_a'(w)|^2 N_{\alpha+2}(w)\, d\lambda(w) + |F_a(\varphi(0))|^2$$

$$= m|a|^2(1-|a|^2)^{\alpha+2}\int |1-\bar{a}w|^{-2(\alpha+3)} N_{\alpha+2}(w)\, d\lambda(w)$$

$$+ |F_a(\varphi(0)|^2,$$

where here, and for the rest of the proof, the letter m denotes a constant which may vary from line to line, but which, at each occurrence, depends only on α.

The last term on the right side of (4) tends to zero as $|a| \to 1-$. Recalling the automorphism ψ_a and its properties from section 5.4, we calculate the important part of the first term on the right:

$$(1-|a|^2)^{\alpha+2}\int |1-\bar{a}w|^{-2(\alpha+3)} N_{\alpha+2}(w)\, d\lambda(w)$$

$$= (1-|a|^2)^{\alpha+2}\int [|\psi_a'(w)|/(1-|a|^2)]^{\alpha+3} N_{\alpha+2}(w)\, d\lambda(w) \qquad [\text{by } 5.4(4)]$$

$$= (1-|a|^2)^{-1}\int |\psi_a'(w)|^{\alpha+1} N_{\alpha+2}(w) |\psi_a'(w)|^2\, d\lambda(w)$$

$$= (1-|a|^2)^{-(\alpha+3)}\int |\psi_a'(\psi_a(z))|^{\alpha+1} N_{\alpha+2}(\psi_a(z))\, d\lambda(z) \quad [w = \psi_a(z), 5.4(6)]$$

$$= (1-|a|^2)^{-1}\int |\psi_a'(z)|^{-(\alpha+1)} N_{\alpha+2}(\psi_a(z))\, d\lambda(z) \qquad [5.4(4) \text{ and } 5.4(5)]$$

$$= (1-|a|^2)^{-(\alpha+2)}\int |1-\bar{a}z|^{2(\alpha+1)} N_{\alpha+2}(\psi_a(z))\, d\lambda(z) \qquad [\text{by } 5.4(5)].$$

Now fix $0 < r < 1$, restrict the last integral to the disc rU, and note that on this disc: $|1 - \bar{a}z| \geq 1 - r$. Upon substituting this inequality into the formula above, and then substituting the result into (4), we obtain:

$$\|C_\varphi F_a\|_{2,\alpha}^2 \geq m|a|^2(1 - |a|^2)^{-(\alpha+2)}(1 - r)^{2(\alpha+1)}\int N_{\alpha+2}(\psi_a(z))\, d\lambda(z).$$

Let $|a| \to 1 -$ in this last inequality, use (1), and then the sub-mean value inequality of Corollary 6.7 (noting, as in section 5.4, that because $\psi_a(\varphi(0))$ tends to ∂U as $|a| \to 1 -$, this is valid for all points a of sufficiently large modulus). The result is:

$$\|C_\varphi\|_e^2 \geq mr^2(1 - r)^{2(\alpha+1)}\sigma_{\alpha+2}(\varphi).$$

Since r is fixed (take $r = \frac{1}{2}$, for example), this is the desired lower estimate. ///

6.9. *Essential norms and angular derivatives.* Recall the notation of Section 3: $\varphi^*(\zeta)$ is the nontangential limit of φ at $\zeta \in \partial U$ (if it exists, while $|\varphi'(\zeta)|$ denotes the magnitude of the angular derivative of φ at ζ, when it exists, and is ∞ when it does not. For $\omega \in \partial U$ and $\gamma \geq 0$ we write:

$$\delta_\gamma(\omega) = \sum\left\{|\varphi'(\zeta)|^{-\gamma} : \zeta \in E(\varphi, \omega)\right\},$$

where

$$E(\varphi, \omega) = \{\zeta \in \partial U, \varphi^*(\zeta) = \omega\}.$$

It follows from the Schwarz lemma that $|\varphi'(z)|$ is bounded away from zero on ∂U by a constant that depends only on $\varphi(0)$. Thus, if $0 < \gamma < \beta$, then by the Julia-Carathéodory theorem, $\delta_\beta \leq \text{const.}\ \delta_\gamma$, where the constant depends only on γ, β, and $\varphi(0)$. By Theorem 3.3, the quantity δ_1 is bounded on the unit circle; hence the same is true of δ_γ for all $\gamma \geq 1$.

We can now present the Bergman space analogues of Theorem 3.3 and 3.5. Observe that the Schwarz lemma guarantees that the quantity on the right side of the inequality below is always finite.

6.10. Theorem. *For $\alpha > -1$, let m denote the constant of Theorem* 6.8, *and write*

$$c(\varphi) = \big(1 + |\varphi(0)|\big)/\big(1 - |\varphi(0)|\big).$$

Then:

$$m \sup\{\delta_{\alpha+2}(\omega) : \omega \in \partial U\} \leq \|C_\varphi\|_{e,\alpha}^2 \leq c(\varphi)\sup\left\{|\varphi'(\zeta)|^{-(\alpha+1)} : \zeta \in \partial U\right\}.$$

Proof. We omit the proof of the lower estimate, which follows from Theorem 6.8 in the same way that Theorem 3.3 followed from Theorem 2.3.

For the upper estimate we use an argument like the one used to prove Theorem 3.5. For each $w \in U \setminus \{0, \varphi(0)\}$, choose $z = z(w) \in \varphi^{-1}\{w\}$ of minimum modulus. Then, letting $h(w) = N_1(w)/(-\log|w|)$, we have

$$(1) \qquad N_{\alpha+2}(w)/(-\log|w|)^{\alpha+2} \leq h(w)[(-\log|z|)/(-\log|w|)]^{\alpha+1}.$$

Now choose $\omega \in \partial U$ and a sequence of points $\{w_n\}$ in U such that $w_n \to \omega$ and $N_{\alpha+2}(w_n)/(-\log|w_n|)^{\alpha+2} \to \sigma_{\alpha+2}(\varphi)$. By passing to a further subsequence, if necessary, we may assume that in addition the sequence of selected preimages $\{z(w_n)\}$ also converges: By the Schwarz lemma its limit must be a point $\zeta \in \partial U$. By part (c) of the Julia-Carathéodory theorem:

$$\overline{\lim}(-\log|z|)/(-\log|\varphi(z)|) = |\varphi'(\zeta)|^{-1} \qquad (z \to \zeta).$$

Hence by (1) above,

$$\sigma_{\alpha+2}(\varphi) \leq \left[\overline{\lim}\, h(w)\right]\left(|\varphi'(\zeta)|^{-1}\right)^{\alpha+1} \qquad (|w| \to 1-)$$
$$= c(\varphi)\left(|\varphi'(\zeta)|^{-1}\right)^{\alpha+1},$$

where the last line follows from the calculation of the upper limit of $h(w)$ done in the proof of Theorem 2.5. The desired result follows from the above inequality and Theorem 6.8. ///

6.11. COROLLARY. *For $\alpha > -1$ the following three conditions on φ are equivalent*:

(a) C_φ *is compact on* A^2_α

(b) $\sigma_{\alpha+2}(\varphi) = 0$.

(c) $|\varphi'(\zeta)| = \infty$ *for every* $\zeta \in \partial U$.

As we pointed out earlier, the equivalence (a) $\Leftrightarrow$ (c) is the main result, for Bergman spaces, of [12]. It also implies the following special case of the Comparison Theorem of [12] (Theorem 5.2): *If C_φ is compact on H^2, then it is compact on A^2_α for every $\alpha > -1$.*

6.12. *Weighted Dirichlet spaces*. The methods of this section also furnish information about the behavior of composition operators on the weighted Dirichlet spaces of the unit disc. These are the spaces D_α, which we consider here only for $\alpha \geq 0$, consisting of functions holomorphic on U with derivative in A^2_α. They are Hilbert spaces in their natural norms, and their size increases with α. D_0 is the classical Dirichlet space, D_1 is H^2, and for $\alpha > 1$, $D_\alpha = A^2_{\alpha+2}$ (see [12], Sections 3 and 5 for more details). If $a < 1$, then D_α is strictly smaller than H^2, and not all composition operators are bounded on D_α. An obvious necessary condition for boundedness is that the inducing function should belong to D_α, but even this is not always enough ([12], Proposition 3.12).

Corollary 6.5 shows that even when $\alpha < 1$, the counting function $N_{\varphi,\alpha}$ determines the norm-related properties of C_φ on D_α. For example, it shows that C_φ is bounded on D_α whenever $\sigma_\alpha(\varphi) < \infty$, and the argument used to obtain the upper estimate of Theorem 6.8 shows that when this is the case, then the essential norm of C_φ on D_α is bounded above by $\sigma_\alpha(\varphi)^{1/2}$. However a similar lower estimate is not yet available to us, since for $\alpha < 1$ we lack an appropriate version of the subharmonic mean value property for $N_{\varphi,\alpha}$ (in fact, for $\alpha = 0$ it is not too difficult to see that the desired lower estimate is *false*).

It was proved in [12] (Theorem 5.3) that if C_φ is bounded on D_α, and if $\beta > \alpha$, then C_φ is bounded on D_β, and compact on that larger space whenever φ satisfies the angular derivative criterion. The proof of the upper estimate of Theorem 6.10 can be easily modified to provide the following result, which is perhaps less general, but certainly more concrete than the one just stated.

6.13. THEOREM. *Suppose $0 \le \alpha < \beta \le 1$, and $\sigma_\alpha(\varphi) < \infty$. Then the essential norm of C_φ on D_β is bounded above by the square root of*

$$[\sigma_\alpha(\varphi)]^\alpha \sup\left\{|\varphi'(\zeta)|^{-(\beta-\alpha)} : \zeta \in \partial U\right\}.$$

In particular, C_φ is compact on D_β whenever φ satisfies the angular derivative criterion.

The case $\beta = 1$ of this result is a generalization, of the sort suggested at the end of Section 3, of Theorem 3.5 and Corollary 3.6.

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