DECOMPOSABILITY AND THE CYCLIC BEHAVIOR OF PARABOLIC COMPOSITION OPERATORS

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Abstract. This paper establishes decomposability for composition operators induced on the Hardy spaces H^p $(1 \le p < \infty)$ by parabolic linear fractional self-maps of the unit disc that are *not* automorphisms. This result, along with a recent theorem of Miller and Miller [15], shows that no such composition operator is supercyclic. The work here completes part of a previous investigation [1] where the author and Paul Bourdon showed that among linear fractional maps of the disc with no interior fixed point, only the parabolic non-automorphisms induce non-hypercyclic composition operators. Additionally it complements results of Robert Smith [19], who proved decomposability in the case of parabolic *automorphisms* and it extends recent work of Gallardo and Montes [6] who, using different methods, established non-supercyclicity for non-automorphisms in the case $p = 2$.

Introduction

This paper deals with parabolic linear fractional mappings φ that take the open unit disc U into itself, and the composition operators C_φ that they induce on the Hardy spaces H^p $(1 \le p < \infty)$ by means of the formula $C_\varphi f = f \circ \varphi$ $(f \in H^p)$. The goal is to show that if φ is *not* an automorphism of U (i.e., if $\varphi(U) \ne U$) then C_φ is: (a) decomposable, and (b) not supercyclic.

To say that an operator T on a Banach space X is *decomposable* means that for every covering of the complex plane $\mathbb{C}$ by a pair $\{V, W\}$ of open sets there is a corresponding pair $\{Y, Z\}$ of closed, T-invariant subspaces such that $X = Y + Z$, the spectrum of $T|_Y$ lies in V, and that of $T|_Z$ lies in W. Decomposability was originally introduced into operator theory in 1963 by Foiaş, but it was not until much later that

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his definition was shown, by several authors independently, to be equivalent the one given here (see [14, Defn. 1.1.1] and the paragraph that precedes it for the appropriate references).

To say that T is *supercyclic* means that there is a vector $x \in X$ such that the *projective orbit* $\{cT^n x : n = 0, 1, 2, \dots \text{ and } c \in \mathbb{C}\}$ is dense in X. Supercyclicity stands midway between the weaker concept of *cyclicity* (some orbit has dense linear span) and *hypercyclicity* (some orbit is dense). The concept was originally introduced by Hilden and Wallen in [10], who showed that it is possessed by every weighted backward shift on ℓ^2 (in particular, even by some quasinilpotent operators!).

The connection between decomposability and supercyclicity was recently established by Miller and Miller, who proved in [15, Theorem 2] (see also [5, Cor. 6.5] and [14, Prop. 3.3.18]) a result that implies:

Theorem M. *Each supercyclic decomposable operator has its spectrum on some (possibly degenerate) circle centered at the origin.*

Now the composition operators treated in this paper have as spectrum either the interval $[0, 1]$ or a spiral that starts at the point 1 and converges to the origin by winding infinitely often around it (see [2, Theorem 6.1, page 102] or §3.10 below). In any case, their spectra do not lie on any circle, hence once these operators are be shown to be decomposable, their non-supercyclicity will follow from Theorem M.

This paper arises from [1], where Paul Bourdon and I classified the cyclic behavior of linear-fractionally induced composition operators on H^2. We showed that among the linear fractional selfmaps of U fixing no point of U (no others have any chance of being hypercyclic [1, Prop. 0.1, page 3]), the only ones *failing* to induce hypercyclic composition operators are the parabolic maps that are not automorphisms. We showed that, nonetheless, such maps induce *cyclic* operators, and wondered if this cyclicity could be improved to *supercyclicity*. In this regard I was able to prove [18] that for such maps φ, the operator C_φ on H^2 had no hypercyclic scalar multiples (clearly any operator with a hypercyclic scalar multiple is supercyclic, but such operators do not exhaust the supercyclic class [12, page 3.4]). Just recently Gallardo and Montes [6] significantly refined the method of [18] to obtain a proof that C_φ is, indeed, not supercyclic on H^2.

The results from [1] discussed above, while phrased only for H^2, hold as well—with almost the same proofs—for any space H^p with $1 \le p < \infty$, so it makes sense to raise in this more general context the supercyclicity question for composition operators induced by parabolic non-automorphisms. The method of [18], although strongly oriented toward Hilbert space, relied in part on Fourier analysis on the real line, and hinted strongly that decomposability might lie at the heart of the supercyclicity issue for the operators in question—a suspicion strongly supported by Theorem M.

Here is an outline of what follows. After a brief survey of prerequisites (Section 1) the study of parabolically induced composition operators will evolve, in Section 2, into a study of translation operators acting on Hardy spaces of the upper half-plane. This will make it possible, in Section 3, to embed each of our parabolic, non-automorphically induced composition operators into a C^2 functional calculus of Fourier integral operators, and from this will follow the desired decomposability and non-supercyclicity.

Even with the appearance of Laursen and Neumann's long-awaited monograph [14], the subject of decomposable operators is still technically formidable. Thus, in the interests of broadening the reach of this paper, I conclude with a couple of purely expository final sections: one devoted to proving that decomposability follows from the existence of a C^∞ functional calculus, and the other to a direct proof that the decomposability and spectral properties of the operators considered here render them non-supercyclic.

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1. Prerequisites

1.1. **Notation.** Throughout this paper p denotes an index which, unless otherwise noted, lies in the interval $[1, \infty)$, and:

- U denotes the open unit disc of the complex plane $\mathbb{C}$,
- ∂U is the unit circle,
- m is Lebesgue arc length measure on ∂U, normalized to have unit mass,
- $L^p(\partial U)$ is the L^p space associated with the measure m, and
- Π_+ denotes the open upper half-plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

1.2. **Hardy spaces.** The *Hardy space* $H^p = H^p(U)$ is the collection of functions f holomorphic on U with

$$\|f\|_p^p := \sup_{0 \le r < 1} \int_{\partial U} |f(r\zeta)|^p \, dm(\zeta) < \infty.$$

The functional $\|\cdot\|_p$ so defined makes H^p into a Banach space. H^∞ is the space of bounded holomorphic functions on U—a Banach space in the norm $\|f\|_\infty := \sup\{|f(z)| : z \in U\}$. Each $f \in H^p$ has, for $[m]$ almost every $\zeta \in \partial U$, a finite radial limit $f^*(\zeta) := \lim_{r\to 1-} f(r\zeta)$, and the map that associates $f \in H^p$ with its *boundary function* f^* is an isometry taking H^p onto the subspace of $L^p(\partial U)$ consisting of functions whose Fourier coefficients of negative index are all zero. The holomorphic function f can be recovered from f^* by either a Cauchy or a Poisson integral.

1.3. **Composition operators.** A holomorphic selfmap of U is just a function that is holomorphic on U and has all its values in U. Each such map φ induces a linear composition operator C_φ on the space of all functions holomorphic on U:

$$C_\varphi f := f \circ \varphi \qquad (f \text{ holomorphic on } U).$$

A classical (and by no means obvious) theorem of Littlewood guarantees that C_φ restricts to a bounded operator on each H^p space, and the study of how the properties of these operators reflect the function theory of their inducing maps has evolved during the past few decades into a lively enterprise; see the monographs [3] and [17] for introductions to the subject, and the conference proceedings [11] for some more recent developments.

1.4. **Parabolic maps.** For linear fractional selfmaps of U the boundedness of C_φ on H^p is elementary; in this paper I consider only a subclass of these maps, the parabolic ones. These are linear fractional transformations that map U into itself and fix exactly one point of the Riemann sphere, a point which must necessarily lie on the unit circle. Each such map is conformally conjugate, via rotation of the unit disc, to one that fixes the point $1 \in \partial U$. Because the composition operators induced by rotations of the disc are isometric isomorphisms of H^p, this rotational conjugation from an arbitrary fixed point on ∂U to fixed point at 1 translates, at the operator level, to an isometric similarity between composition operators. Because all of the operator-theoretic phenomena to be considered in this paper are similarity-invariant, nothing will therefore be lost by always placing the fixed point of φ at 1.

Suppose, then, that φ is a parabolic selfmap of U with $\varphi(1) = 1$. The map τ defined by

$$\tau(z) = i\,\frac{1+z}{1-z} \qquad z \in \mathbb{C}\backslash\{1\} \tag{1}$$

maps the unit disc conformally onto the upper half-plane $\Pi_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, takes $\partial U \backslash \{1\}$ homeomorphically onto the real line, and sends the point 1 to ∞. The map $\Phi := \tau \circ \varphi \circ \tau^{-1}$ is therefore a linear fractional map that takes Π_+ into itself and fixes ∞, hence it must be translation by some $a \in \mathbb{C}$ with $\operatorname{Im} a \geq 0$, that is, $\Phi(w) = w + a$ for $w \in \mathbb{C}$. Let us call a the *translation parameter* of both the translation Φ of Π_+ and the original parabolic mapping φ of U. Note that φ is an automorphism of U precisely when Φ has the same property on Π_+, and this happens if and only if the translation parameter is real.

This characterization of parabolic composition operators suggests that they may be studied most effectively by shifting attention from the unit disc to the upper half-plane; I develop this point of view in the next section.

2. Migrating to the Upper Half-Plane

2.1. **Hardy spaces on the upper half-plane.** There are two ways to define a Hardy space H^p for the upper half-plane:

(a) $H^p(\Pi_+)$ is the space of functions F holomorphic on Π_+ with $F \circ \tau \in H^p(U)$. The norm $\|\cdot\|_p$ defined on $H^p(\Pi_+)$ by $\|F\|_p := \|F \circ \tau\|_p$ (where the norm on the right is the one for $H^p(U)$) makes $H^p(\Pi_+)$ into a Banach space, and insures that the map $C_\tau : H^p(\Pi_+) \to H^p(U)$ is an isometry taking $H^p(\Pi_+)$ onto $H^p(U)$. In particular, for each $F \in H^p(\Pi_+)$ the "radial limit" $F^*(x) = \lim_{y\to 0} F(x+iy)$ exists for a.e. $x \in \mathbb{R}$, and a change of variable involving the map τ shows that the norm of F can be computed by integrating over $\mathbb{R}$:

$$\|F\|_p^p = \frac{1}{\pi}\int_{-\infty}^{\infty} |F^*(x)|^p \, \frac{dx}{1+x^2}.$$

(b) $\mathcal{H}^p(\Pi_+)$ is the space of functions F holomorphic on Π_+ for which

$$\|F\|_p^p := \sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|^p \, dx < \infty.$$

Once again the norm defined on the space (which, although denoted by the same symbol as the previous norms, is different from them) makes it into a Banach space.

These two spaces are not the same; the map C_τ takes $\mathcal{H}^p(\Pi_+)$ onto the dense subspace $(1-z)^{2/p}H^p(U)$ of $H^p(U)$, hence $\mathcal{H}^p(\Pi_+)$ is a dense subspace of $H^p(\Pi_+)$. Finally, the norm in $\mathcal{H}^p(\Pi_+)$ can be computed on the boundary: $\|F\|_p^p = \int_{-\infty}^{\infty} |F^*(x)|^p \, dx$, so that $\mathcal{H}^p(\Pi_+)$ can be regarded as a closed subspace of $L^p(\mathbb{R})$. For $p = 1$ it is the subspace consisting of functions whose Fourier transforms vanish on $(-\infty, 0]$, and a similar interpretation can be made for $1 < p \le 2$. For a detailed exposition of these and other basic facts about Hardy spaces in half-planes I refer the reader to [7, Chapter II], [8, Chapter 8], or [13, Chapter VI].

2.2. **Eigenvalues of $\mathbf{C}_\varphi$.** Suppose φ is a parabolic selfmap of U with fixed point at 1, and let $a \in \mathbb{C}$ with $\operatorname{Im} a \ge 0$ be its translation parameter, so that $\Phi = \tau \circ \varphi \circ \tau^{-1}$ is just "translation by a" in Π_+. For $t \ge 0$ let $E_t(w) = e^{itw}$ for $w \in \Pi_+$. E_t is a bounded holomorphic function on Π_+, hence

$$e_t(z) = E_t(\tau(z)) = \exp\left\{-t\,\frac{1+z}{1-z}\right\} \qquad (z \in U),$$

defines bounded holomorphic function on U (the t-th power of the *unit singular function*). Because of this boundedness $e_t \in H^p(U)$, or equivalently, $E_t \in H^p(\Pi_+)$ for each $1 \le p \le \infty$. Furthermore $C_\Phi E_t = e^{iat} E_t$ hence also $C_\varphi e_t = e^{iat} e_t$ for each $t \ge 0$. Thus for each such t the function e_t is an eigenvector of $C_\varphi : H^p(U) \to H^p(U)$ with corresponding eigenvalue e^{iat}. Thus $\Gamma_a := \{e^{iat} : t \ge 0\}$ is a subset of the spectrum of C_φ. If a is real, so that φ is an automorphism, then Γ_a covers the unit circle infinitely often, and it turns out that ∂U is precisely the spectrum of C_φ, a result proved over thirty years ago by Nordgren [16]. If $\operatorname{Im} a > 0$ then φ is not an automorphism, and Γ_a is a curve that starts at 1 when $t = 0$ and converges to 0 as $t \to \infty$. If a is pure imaginary then $\Gamma_a = (0, 1]$, otherwise Γ_a spirals infinitely often around the origin, converging to the origin with strictly decreasing modulus. Thus in these non-automorphic cases the spectrum of C_φ contains $\Gamma_a \cup \{0\}$, and it is a (special case of a) result of Cowen [2, Theorem 6.1] that $\Gamma_a \cup \{0\}$ is indeed the whole spectrum. I will give an alternate proof of this fact in Section 3.

2.3. **$\mathbf{C}_\Phi$ as a convolution operator.** We saw in §1.4 that each parabolic selfmap φ of U that fixes the point 1 has the representation $\varphi = \tau^{-1} \circ \Phi \circ \tau$, where τ is the linear fractional mapping of U onto Π_+ given by (1), and Φ is the mapping of translation by some fixed vector a in the closed upper half-plane: $\Phi(w) = w + a$ for $w \in \Pi_+$. At the operator level this conjugacy turns into the similarity $C_\varphi = C_\tau C_\Phi C_\tau^{-1}$, where C_τ is an isometry mapping $H^p(\Pi_+)$ into $H^p(U)$, and C_Φ is a bounded operator on $H^p(\Pi_+)$.

Since all the operator theoretic phenomena being investigated here are preserved by similarity, nothing will be lost (in fact much will be gained) by shifting attention from C_φ on $H^p(U)$ to C_Φ on $H^p(\Pi_+)$. The advantage here is that when the original parabolic mapping φ of U is not an automorphism, the operator C_Φ on $H^p(\Pi_+)$ can be represented as a convolution operator. The key is that each $F \in H^p(\Pi_+)$ is the Poisson integral of its boundary function:

$$F(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\, F^*(t)\,dt \qquad (x+iy \in \Pi_+).$$

Since φ is not an automorphism, its translation parameter $a = \alpha + i\beta$ lies in the (open) upper half-plane, and $C_\Phi F(w) = F(w+a)$ for $w \in \Pi_+$. Thus for each $F \in H^p(\Pi_+)$ and $x \in \mathbb{R}$:

$$(2) \qquad (C_\Phi F)^*(x) = F(x+\alpha+i\beta) = \int_{-\infty}^{\infty} P_a(x-t)F^*(t)\,dt$$

where

$$(3) \qquad P_a(x) = \frac{1}{\pi}\,\frac{\beta}{(x-\alpha)^2+\beta^2}$$

is the (upper half-plane) Poisson kernel for the point $a \in \Pi_+$.

From now on I will drop the superscript "$*$" that distinguished holomorphic functions from their radial limit functions, and simply regard each function $F \in H^p(\Pi_+)$ to be either a holomorphic function on the upper half-plane, or the associated radial limit function—an element of the space $L^p(\mu)$, where μ is the *Cauchy measure*

$$(4) \qquad d\mu(x) := \frac{1}{\pi}\,\frac{dx}{1+x^2}\ ,$$

a Borel probability measure on $\mathbb{R}$. In each case, either the context or an explicit statement will make clear which interpretation of F is intended.

Correspondingly, the operator C_Φ can now be given two different interpretations: either as the original composition operator on holomorphic functions, or—by (2) and (3) above—as the restriction to $H^p(\Pi_+)$-boundary functions of the convolution operator

$$(5) \qquad C_\Phi F = F * P_a \qquad (F \in L^p(\mu)).$$

From this convolution representation arises the functional calculus which lies at the heart of this paper.

3. A functional calculus

The goal of this section is to prove:

3.1. **Theorem.** *If φ is a parabolic linear fractional selfmap of U that is not an automorphism, then $C_\varphi : H^p \to H^p$ has a C^2 functional calculus.*

For our purposes the conclusion means that there is an algebra homomorphism $\gamma \to \gamma(C_\varphi)$ from $C^2(\mathbb{C})$ into $\mathcal{L}(H^p)$, the algebra of bounded linear operators on H^p, such that if β and γ belong to C^2, then (letting $\sigma(C_\varphi)$ denote the spectrum of C_φ):

(FC1) If $\beta \equiv \gamma$ on $\sigma(C_\varphi)$ then $\beta(C_\varphi) = \gamma(C_\varphi)$,

(FC2) If $\gamma(z) \equiv z$ on $\sigma(C_\varphi)$ then $\gamma(C_\varphi) = C_\varphi$, and

(FC3) If $\gamma \equiv 1$ on $\sigma(C_\varphi)$ then $\gamma(C_\varphi)$ is the identity operator on H^p.

From this functional calculus will follow the decomposability and, therefore the non-supercyclicity, of C_φ. It turns out that for any operator on a Banach space, properties (FC1)–(FC3) follow from the weaker assumption that (FC2) and (FC3) hold with the spectrum replaced by the whole complex plane (see [14, Theorem 1.4.10]). However for the functional calculus constructed here, the full strength of (FC1)–(FC3) will be immediately apparent.

Because the existence of a functional calculus is similarity invariant, it will be enough to carry out the construction in $H^p(\Pi_+)$, with the translation operator C_Φ on that space standing in for C_φ. The work of this section takes place exclusively on the boundary, so that $H^p(\Pi_+)$ will be interpreted as a subspace of $L^p(\mu)$, where μ is the Cauchy probability measure on $\mathbb{R}$ given by (4).

We construct our functional calculus by using (5) to view C_Φ as a convolution operator on $L^p(\mu)$, and then restricting to the invariant subspace $H^p(\Pi_+)$. The following well known sufficient condition is the key to proving boundedness for the operators in question. Even though it is stated here only for the Cauchy measure μ on the Borel sets of $\mathbb{R}$, it is valid for any positive measure on any measure space.

3.2. **The Schur Test** [4, Page 518, Problem 54]. *Suppose K is a non-negative Borel measurable function on $\mathbb{R}^2$, and that there exists a positive, finite constant C such that:*

(a) $\displaystyle\int K(x,y)\, d\mu(y) \leq C \quad$ *for a.e.* $x \in \mathbb{R}$,

(b) $\displaystyle\int K(x,y)\,d\mu(x) \le C$ *for a.e.* $y \in \mathbb{R}$.

For each non-negative Borel function f *on* $\mathbb{R}$ *define*

$$T_K f(x) := \int K(x,y) f(y)\,d\mu(y) \qquad (x \in \mathbb{R}). \tag{6}$$

Then $\|T_K f\|_p \le C\|f\|_p$ *for each* $1 \le p \le \infty$*; in particular* T_K *can now be defined by (6) on all of* $L^p(\mu)$*, where it acts as a bounded linear operator.*

The Schur Test yields the following criterion for a convolution operator to be bounded on $L^p(\mu)$.

3.3. **Proposition.** *Suppose* $k : \mathbb{R} \to \mathbb{C}$ *is a bounded Borel measurable function such that*

$$|k(x)| = O(|x|^{-2}) \quad \textit{as } |x| \to \infty. \tag{7}$$

Then the mapping $f \to k * f$ *is a bounded linear operator on* $L^p(\mu)$ *for each* $1 \le p \le \infty$.

Proof. The decay condition (7) insures that there is no difficulty in convolving k with any function in $L^p(\mu)$. To apply the Schur test let

$$K(x,y) = \pi(1+y^2)k(x-y) \qquad (x, y \in \mathbb{R}),$$

so that for $x \in \mathbb{R}$:

$$k * f(x) := \int k(x-y) f(y)\,dy = \int K(x,y) f(y)\,d\mu(y)$$

(unadorned integral signs now refer to integration over the entire real line). So to prove the boundedness of the convolution operator it suffices to show that $|K|$ satisfies the hypotheses of Schur's Test. Hypothesis (a) is easy; for each $x \in \mathbb{R}$:

$$\int |K(x,y)|\,d\mu(y) = \int |k(x-y)|\,dy = \int |k(y)|\,dy < \infty$$

where the integrability of k follows from the decay condition (7).

For hypothesis (b) note that for every $y \in \mathbb{R}$:

$$\begin{aligned}\int |K(x,y)|\,d\mu(x) &= (1+y^2)\int |k(x-y)|\,\frac{dx}{1+x^2} \\ &\le C(1+y^2)\int \frac{1}{1+(x-y)^2}\cdot\frac{1}{1+x^2}\,dx\ ,\end{aligned}$$

where the inequality arises from (7), with C independent of y. The last integral in this display is, by (3) above, a constant multiple of $P_i * P_i$, the convolution of the Poisson kernel for the point $i \in \Pi_+$ with itself. Now this convolution square is just the Poisson kernel for the point $2i$, namely $(2/\pi)(4+y^2)^{-1}$ (see the next paragraph for details). This establishes the boundedness of $\int |K(x,y)|\,d\mu(y)$, and with it, that of the operator of convolution by k on $L^p(\mu)$. $\square$

3.4. **Remark on the Poisson kernel.** The identity $P_i * P_i = P_{2i}$ used in the proof of Proposition 3.3 is a special case of the semigroup identity

$$P_a * P_b = P_{a+b} \qquad (a, b \in \Pi_+), \tag{8}$$

which one can prove using either the Fourier transform or the Poisson integral representation of harmonic functions. Since the Fourier transform of the Poisson kernel will play a crucial role in the sequel, I'd like to take a moment to show how it leads to (8).

The Fourier transform of P_i is well known; it is

$$\widehat{P_i}(\lambda) := \frac{1}{\pi}\int \frac{e^{-i\lambda t}}{1+t^2}\,dt = e^{-|\lambda|} \qquad (\lambda \in \mathbb{R}). \tag{9}$$

Now $P_a(t) = \beta^{-1}P_i((t-\alpha)/\beta)$ for each $a = \alpha + i\beta \in \Pi_+$, so it follows from (9) and a change of variable that:

$$\widehat{P_a}(\lambda) = e^{-i\alpha\lambda}e^{-|\lambda|\beta} = \begin{cases} e^{-i\overline{a}\lambda} & (\lambda \geq 0) \\ e^{-ia\lambda} & (\lambda \leq 0) \end{cases} \tag{10}$$

from which it follows easily that for $a, b \in \Pi_+$, $\widehat{P_a * P_b} = \widehat{P_a}\,\widehat{P_b} = \widehat{P_{a+b}}$ at every point of $\mathbb{R}$. This, by the uniqueness of Fourier transforms, implies the desired semigroup property. $\square$

Proposition 3.3 provides the foundation for the next result, which is the major building block in the construction of our functional calculus. For all that follows we fix $a = \alpha + i\beta \in \Pi_+$.

3.5. **Proposition.** *Suppose $\gamma \in C^2(\mathbb{C})$ with $\gamma(0) = \gamma(1) = 0$. Let k_γ be the inverse Fourier transform of $\gamma \circ \widehat{P_a}$. Then the convolution operator $f \to k_\gamma * f$ is bounded on $L^p(\mu)$ and maps $H^p(\Pi_+)$ into itself.*

Proof. It follows from (10) that $|\widehat{P_a}(\lambda)| = e^{-\beta|\lambda|}$ for each $\lambda \in \mathbb{R}$. Because γ is differentiable and vanishes at 0, the composition $\gamma \circ \widehat{P_a}$ inherits the exponential decay of $\widehat{P_a}$ at $\pm\infty$, and is therefore integrable, hence there is no problem in defining k_γ, its inverse Fourier transform. In fact the first and second derivatives of $\gamma \circ \widehat{P_a}$ on both half-intervals $(0,\infty)$ and $(-\infty, 0)$ have the same exponential decay, and thus k_γ can be estimated by splitting its defining Fourier integral into two pieces—one over each half-interval—and integrating the results twice by parts, using the condition $\gamma(0) = \gamma(1) = 0$ to get rid of the boundary terms at the first stage. The result is that the asymptotic estimate (7) is valid for k_γ, hence by Proposition 3.3 the associated convolution operator is bounded.

As for H^p-preservation, observe first that $\mathcal{H}^p(\Pi_+)$ is a dense subspace of $H^p(\Pi_+)$. One way to see this is to note that C_τ takes $\mathcal{H}^p(\Pi_+)$ to $(1-z)^{2/p}H^p(U)$ (see [7, Lemma 1.2, page 51] or [8, page 130]), and (by definition) $H^p(\Pi_+)$ to $H^p(U)$. An application of Beurling's theorem then seals the argument. Now the functions in $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ whose Fourier transforms vanish on the negative real axis form a dense subspace of $\mathcal{H}^p(\Pi_+)$, and therefore of $H^p(\Pi_+)$, so it is enough to prove that $k_\gamma * f \in \mathcal{H}^p(\Pi_+)$ for each such function f. Clearly this convolution lies in $L^p(\mathbb{R}) \cap L^1(\mathbb{R})$, and its Fourier transform, which is $\widehat{k_\gamma} \cdot \hat{f}$, vanishes where $\hat{f}$ does—on the negative real axis. Thus $k_\gamma * f \in \mathcal{H}^p(\Pi_+) \subset H^p(\Pi_+)$ and the proof is complete. □

3.6. **The functional calculus for $\mathbf{C_\Phi}$ on $\mathbf{L^p}(\mu)$.** As usual, we denote by Φ the mapping of "translation by $a \in \Pi_+$" on $\mathbb{C}$.

Let $\mathcal{G}$ denote the class of functions γ that satisfy the hypotheses of Proposition 3.5—twice continuously differentiable on $\mathbb{C}$ and vanishing at both 0 and 1. For $\gamma \in \mathcal{G}$ define $\gamma(C_\Phi)$ to be the operator of convolution with k_γ, acting on $L^p(\mu)$. According to the work just completed, $\gamma(C_\Phi)$ is a bounded operator on $L^p(\mu)$ that leaves invariant the closed subspace $H^p(\Pi_+)$ (still being viewed as a space of functions on the real line).

If γ_1 and γ_2 belong to $\mathcal{G}$ and coincide on $\widehat{P_a}(\mathbb{R})$, then so do their left compositions with $\widehat{P_a}$, and hence so do the inverse Fourier transforms of these compositions. Since these inverse Fourier transforms are just the convolution kernels k_{γ_1} and k_{γ_2}, it follows that $\gamma_1(C_\Phi) = \gamma_2(C_\Phi)$.

The map $\gamma \to \gamma(C_\Phi)$ is clearly additive and homogeneous with respect to scalar multiplication. To see that it is also multiplicative, let γ_1 and γ_2 belong to $\mathcal{G}$ and observe that

$$\widehat{k_{\gamma_1\cdot\gamma_2}} = (\gamma_1 \cdot \gamma_2) \circ \widehat{P_a} = (\gamma_1 \circ \widehat{P_a}) \cdot (\gamma_2 \circ \widehat{P_a}) = \widehat{k_{\gamma_1}} \cdot \widehat{k_{\gamma_2}},$$

hence $k_{\gamma_1\cdot\gamma_2} = k_{\gamma_1} * k_{\gamma_2}$. It follows that for each $f \in L^p(\mu) \cap L^1(\mathbb{R})$ (a dense subspace of $L^p(\mu)$):

$$(\gamma_1 \cdot \gamma_2)(C_\Phi)f := k_{\gamma_1\cdot\gamma_2} * f = k_{\gamma_1} * (k_{\gamma_2} * f) = \gamma_1(C_\Phi)[\gamma_2(C_\Phi)f],$$

which establishes the desired multiplicative property. □

The arguments so far have shown that the map $\gamma \to \gamma(C_\Phi)$ is an algebra homomorphism of $\mathcal{G}$ into $\mathcal{L}(L^p(\mu))$. It remains to extend this map appropriately to all of $C^2(\mathbb{C})$. For this it suffices to note that each $\gamma \in C^2(\mathbb{C})$ can be written uniquely as

$$\gamma(z) = a + bz + \gamma_0(z) \qquad (z \in \mathbb{C}),$$

where $a = \gamma(0)$, $b = \gamma(1) - \gamma(0)$, and $\gamma_0 \in \mathcal{G}$. Thus

$$\gamma(C_\Phi) := aI + bC_\Phi + \gamma_0(C_\Phi)$$

defines a bounded linear operator on $L^p(\mu)$ that takes $H^p(\Pi_+)$ into itself. One checks easily that the homomorphic property previously noted on $\mathcal{G}$ for the mapping $\gamma \to \gamma(C_\Phi)$ carries over to the extension just defined on $C^2(\mathbb{C})$, and that this extension has all the properties needed to be a functional calculus for C_Φ on $L^p(\mu)$, except that the uniqueness conditions (FC1)–(FC3), which are supposed to hold for $\sigma(C_\Phi)$, have been proven instead for $\widehat{P_a}(\mathbb{R})$. The next result shows that (FC1)–(FC3) hold just as advertised.

3.7. **Proposition.** $\sigma(C_\Phi : L^p(\mu) \to L^p(\mu)) = \widehat{P_a}(\mathbb{R}) \cup \{0\}$.

Proof. For $t \in \mathbb{R}$ let $E_t(x) = e^{itx}$ $(x \in \mathbb{R})$. Since these functions are continuous and bounded (in fact, unimodular) on $\mathbb{R}$, they all belong to $L^p(\mu)$. For $t \geq 0$ these functions turned out to be eigenvectors of $C_\Phi : H^p(\Pi_+) \to H^p(\Pi_+)$. The first order of business is to show that the full collection serves as eigenvectors for C_Φ on $L^p(\mu)$. For this, fix x and t in $\mathbb{R}$ and note that:

$$\begin{aligned} C_\Phi(E_t(x)) &:= P_a * E_t(x) = \int e^{it(x-\xi)} P_a(\xi)\, d\xi \\ &= e^{itx}\widehat{P_a}(t) = \widehat{P_a}(t) E_t(x) \end{aligned}$$

so $\widehat{P_a}(t)$ is an eigenvalue of $C_\Phi : L^p(\mu) \to L^p(\mu)$ corresponding to the eigenvector E_t. Thus $\widehat{P_a}(\mathbb{R})$ is contained in the $L^p(\mu)$ spectrum of C_Φ, hence so is its closure $\widehat{P_a}(\mathbb{R}) \cup \{0\}$.

To complete the proof it suffices to show that $\lambda \notin \widehat{P_a}(\mathbb{R}) \cup \{0\}$ implies $\lambda \notin \sigma(C_\Phi)$. For each such λ there exists a function $\gamma \in C^2(\mathbb{C})$ with $\gamma(z) = (z-\lambda)^{-1}$ for $z \in \widehat{P_a}(\mathbb{R})$. Now $\psi(z) = z - \lambda$ is also a C^2 function on $\mathbb{C}$, and $\psi \cdot \gamma \equiv 1$ on $\widehat{P_a}(\mathbb{R})$. Thus by the properties derived so far for our functional calculus:

$$\gamma(C_\Phi)(C_\Phi - \lambda I) = \gamma(C_\Phi)\psi(C_\Phi) = (\gamma \cdot \psi)(C_\Phi) = I,$$

where I is the identity map on $L^p(\mu)$. This display shows, because all the operator factors therein commute, that $C_\Phi - \lambda I$ is invertible on $L^p(\mu)$, hence $\lambda \notin \sigma(C_\Phi)$. □

3.8. **Remark.** Recall from §2.2 our observation that the set

$$\Gamma_a := \widehat{P_a}([0,\infty)) = \{e^{iat} : t \geq 0\},$$

is a curve that spirals from the point 1 asymptotically into the origin. By formula (10), $\widehat{P_a}(\mathbb{R})$ is the union of Γ_a and its reflection in the x-axis, a double-spiral joining the point 1 to the origin.

It remains only to check that the functional calculus constructed above for C_Φ on $L^p(\mu)$ restricts properly to the subspace $H^p(\Pi_+)$, which we have already seen is invariant for all the operators involved (Proposition 3.5). This is the content of the next two results.

3.9. **Proposition.** *Suppose $\gamma_1, \gamma_2 \in C^2(\mathbb{C})$ with $\gamma_1 \equiv \gamma_2$ on Γ_a. Then $\gamma_1(C_\Phi) = \gamma_2(C_\Phi)$ on $H^p(\Pi_+)$.*

Proof. It is enough to prove that the two operators coincide on the dense subspace $H^p(\Pi_+) \cap L^1(\mathbb{R})$ of $H^p(\Pi_+)$. For f in this subspace the Fourier transform $\hat{f}$ vanishes on the negative real axis. Our hypothesis guarantees that $\gamma_1 \circ \widehat{P_a} = \gamma_2 \circ \widehat{P_a}$ on $[0,\infty)$, so at each point of $\mathbb{R}$ we have:

$$\begin{aligned}
[\gamma_1(C_\Phi)f]\hat{} &= [k_{\gamma_1} * f]\hat{} = \widehat{k_{\gamma_1}} \cdot \hat{f} = (\gamma_1 \circ \widehat{P_a}) \cdot \hat{f} \\
&= (\gamma_2 \circ \widehat{P_a}) \cdot \hat{f} = [\gamma_2(C_\Phi)f]\hat{}
\end{aligned}$$

hence $\gamma_1(C_\Phi)f = \gamma_2(C_\Phi)f$.

3.10. **Corollary.** $\sigma(C_\Phi : H^p(\Pi_+) \to H^p(\Pi_+)) = \Gamma_a \cup 0$.

Proof. We have already seen that each $\lambda \in \Gamma_a$ is an eigenvalue of $C_\Phi : H^p(\Pi_+) \to H^p(\Pi_+)$, so the spectrum of this operator contains $\Gamma_a \cup \{0\}$. To go the other way it is enough to show that if $\lambda \notin \Gamma_a \cup \{0\}$ then λ is not in the spectrum, i.e. that $C_\Phi - \lambda I$ is invertible on $H^p(\Pi_+)$. Now the hypothesis on λ is that $z - \lambda$ is bounded away from zero on Γ_a, hence there exists $\gamma \in C^2(\mathbb{C})$ with $\gamma(z) = (z-\lambda)^{-1}$ on Γ_a. Since $(z-\lambda)\gamma(z) \equiv 1$ on Γ_a it follows from Proposition 3.9 that, just as in the proof of Proposition 3.7, the operator $\gamma(C_\Phi)$ is the inverse, on $H^p(\Pi_+)$, of $C_\Phi - \lambda I$. □

4. Decomposability

As promised in the Introduction, I include these final two sections entirely for the convenience of the reader. While there may be some originality in the organization of Section 5, the material in this section comes right out of [14, Theorem 1.4.10].

In the last section we constructed, for each composition operator induced on H^p by a parabolic non-automorphism, a C^2-functional calculus. The point of this section is that every Banach space operator with even a C^∞ functional calculus is decomposable.

So assume that X is a Banach space and T a bounded linear operator on X, and that T has a C^∞ functional calculus in the sense of the discussion following Theorem 3.1.

To each compact subset K of $\mathbb{C}$ let us attach the subspace $E(K)$ of X formed by intersecting the null spaces of all the operators $\eta(T)$ where $\eta \in C^\infty(\mathbb{C})$ and $K \cap \operatorname{spt} \eta = \emptyset$. Everything depends on the following result.

4.1. **Lemma.** *For each compact subset K of $\mathbb{C}$, the subspace $E(K)$ is closed and $T-$ invariant, with $\sigma(T|_{E(K)}) \subset K$. Moreover, if λ is an eigenvalue of T then the following are equivalent:*

(a) $\lambda \in K$.
(b) *Every λ-eigenvector of T lies in $E(K)$.*
(c) *Some λ-eigenvector of T lies in $E(K)$.*

Proof. That $E(K)$ is closed and T-invariant is routine, so I omit the argument. For the spectral inclusion, suppose $\lambda \in \mathbb{C}\backslash K$. We wish to show that $T - \lambda I$ is invertible on $E(K)$. Choose an open set V that contains K but whose closure does not contain λ, and observe that there is a C^∞ function η on the plane with $\eta(z) = (z-\lambda)^{-1}$ on V. Thus $\gamma(z) := (z-\lambda)\eta(z)$ is C^∞ on the plane, and $\equiv 1$ on V, and so $1-\gamma$ has support disjoint from K. Therefore if $x \in E(K)$ we have (by the definition of $E(K)$) $(1-\gamma)(T)x = 0$, hence:

$$I = \gamma(T) = (T-\lambda I)\eta(T) = \eta(T)(T-\lambda I) \quad \text{on } E(K),$$

which establishes the desired invertibility.

As for eigenvalues and eigenfunctions, note first that if λ is an eigenvalue and x an eigenvector for λ then it is easy to check that x is a $\gamma(\lambda)$-eigenvector for $\gamma(T)$ for any $\gamma \in C^\infty(\mathbb{C})$. The equivalence of (a), (b), and (c) follows easily from this and the fact that $\lambda \in K$ if and only if $\gamma(\lambda) = 0$ for every $\gamma \in C^\infty(\mathbb{C})$ with support disjoint from K. $\square$

4.2. **Theorem.** *Suppose X is a Banach space and $T \in \mathcal{L}(X)$ has a C^∞ functional calculus in the sense of §3.1. Then T is decomposable.*

Proof. Suppose V and W are nonvoid open subsets that cover the plane. Recall from the Introduction that the goal is to find closed T-invariant subspaces Y and Z whose sum is X such that the restrictions of T to Y and Z have spectra that lie, respectively, in U and V.

To make the decomposition, let $\{\beta, \gamma\}$ be a C^∞ partition of unity on $\sigma(T)$ subordinate to the open covering $\{U, V\}$. Because $\beta + \gamma \equiv 1$ on $\sigma(T)$, the operator $\beta(T) + \gamma(T)$ is the identity on X. Thus

$$\beta(T)X + \gamma(T)X = X.$$

Let $Y = E(\operatorname{spt}\beta)$ and $Z = E(\operatorname{spt}\gamma)$. Then the spectral inclusions follow immediately from Lemma 4.1. To see that $X = Y + Z$ just note that if x is in the range of $\beta(T)$, say $x = \beta(T)x'$ for some $x' \in X$, and if $\eta \in C^\infty(\mathbb{C})$ has support disjoint from that of β, then $\eta \cdot \beta \equiv 0$, so

$$0 = (\eta \cdot \beta)(T)x' = \eta(T)\beta(T)x' = \eta(T)x$$

hence $x \in Y$. In other words $\operatorname{ran}\beta(T) \subset Y$, and similarly $\operatorname{ran}\gamma(T) \subset Z$. Since, as noted above, X is the sum of the smaller subspaces, it is also the sum of the larger ones. $\square$

4.3. **Remarks.** (a) Operators with a C^∞ functional calculus are called *generalized scalar operators*. These form a proper subclass of the decomposable operators (see the discussion following Theorem 1.4.10 of [14] for references).

(b) If T is a Banach space operator whose spectrum lies in the unit circle, and for which there is a positive integer N such

$$\|T^k\| = \mathrm{O}(|k|^N) \qquad (|k| \to \infty), \tag{11}$$

then a C^∞ functional calculus can be constructed for T by setting $\gamma(T) := \sum_{-\infty}^{\infty} \hat{\gamma}(k)T^k$, where for $\gamma \in C^\infty(\mathbb{C})$ and $\hat{\gamma}(k)$ is the k-th Fourier coefficient of the restriction of γ to the unit circle. If φ is a parabolic *automorphism* of U then it is well known that $T = C_\varphi$ obeys (11) (see [16], for example), and is therefore—as was first noted by Smith in [19]—decomposable. In contrast to the operators we have been considering here, these automorphically-induced composition operators C_φ *are* supercyclic; in fact *hypercyclic* [1, Thm. 2.2, page 25].

5. (Non)Supercyclicity

So far we have seen that for $1 \leq p < \infty$, composition operators induced on H^p by parabolic non-automorphisms of U are decomposable, and that no such map has its spectrum lying on a circle. As previously mentioned, Theorem M of the Introduction then asserts that no such operator can be supercyclic.

Because the proof of Theorem M requires considerable background, I include for the reader's convenience this final section, which provides a mostly self-contained proof of non-supercyclicity for the class of composition operators we are considering here. The key to the argument is the following result, which occurs in [5, Theorem 6.1] and [6, Prop. 2.1].

5.1. **Lemma.** *A bounded linear operator T on a Banach space X is* not *supercyclic on X whenever its spectrum can be split into a disjoint union of nonvoid compact sets K_1 and K_2, where $K_1 \subset \{|z| < r\}$ and $K_2 \subset \{|z| > r\}$ for some positive r.*

Proof. An operator is supercyclic if and only if every one of its non-zero scalar multiples is supercyclic, so we may, without loss of generality, assume that $r = 1$. The Riesz functional calculus provides a direct sum decomposition $X = X_1 \oplus X_2$ where X_i is a closed, T-invariant subspace of X and $\sigma(T|_{X_i}) \subset K_i$ $(i = 1, 2)$. Because the spectrum of $T|_{X_1}$ lies in the open unit disc, the spectral radius formula implies that the positive powers of this operator converge to zero in the operator norm. Similarly, the spectrum of the restriction of T to X_2 lies outside the closed unit disc, hence by an easy argument, $\|T^n x\| \to \infty$ for every $0 \neq x \in X_2$ (see, e.g., [5, Lemma 6.3] for details).

Now suppose $0 \neq x \in X$. The goal is to show that x is not a supercyclic vector. In the decomposition $x = x_1 + x_2$ with $x_i \in X_i$ $(i = 1, 2)$ this will be trivial if either x_1 or x_2 is the zero vector. So suppose otherwise, in which case the Hahn-Banach theorem provides a bounded linear functional Λ on X that vanishes identically on X_2, but has $\Lambda(x_1) \neq 0$. Let y be a non-zero vector that is a limit point of the projective T-orbit of x, so there exist a sequence $\{c_j\}$ of scalars and a strictly increasing sequence $\{n_j\}$ of non-negative integers such that $c_j T^{n_j} \to y$. Therefore:

$$\frac{|\Lambda(y)|}{\|y\|} = \lim_j \frac{|\Lambda(c_j T^{n_j} x)|}{\|c_j T^{n_j} x\|} = \lim_j \frac{|\Lambda(T^{n_j} x_1)|}{\|T^{n_j}(x_1 + x_2)\|} \ .$$

In the last fraction the numerator is bounded above by $\|\Lambda\| \, \|T^{n_j} x_1\|$, which converges to zero, while the denominator is bounded below by $\|T^{n_j} x_2\| - \|T^{n_j} x_1\|$, which converges to ∞. Thus the fraction itself converges to zero, and so $\Lambda(y) = 0$.

This shows that for any $0 \neq x \in X$ there is a nontrivial bounded linear functional Λ on X such that each limit point of the projective T-orbit of x lies in the null space of Λ. This projective orbit is therefore not dense in X so, as desired, x is not a supercyclic vector for T. □

5.2. **Remark.** This result lies at the heart of the proof that for any supercyclic operator T there must exist a (possibly degenerate) circle centered at the origin that intersects every component of the spectrum of T (see [9, Proposition 3.1] [5, Theorem 6.1] or [6, Prop. 2.1]). This "circle theorem" can, in turn, be considered an extension of a result of Kitai, who proved in [12, Theorem 2.8] that every component of the spectrum of a *hypercyclic* operator must intersect the unit circle. I thank Alfonso Montes for pointing out the reference to Herrero's paper.

5.3. **Restriction and quotient maps.** If T is a bounded linear operator on X, then any closed, T-invariant subspace Y of X gives rise to two further operators: the usual *restriction operator* $T|_Y : Y \to Y$ and the perhaps less familiar *quotient operator* $T/Y : X/Y \to X/Y$, defined by:

$$(T/Y)(x+Y) = Tx + Y \qquad (x \in X).$$

Let $\sigma_f(T)$ denote the union of $\sigma(T)$ with all the bounded components of its complement, the so-called *full spectrum* of T. It is well known that $\sigma(T|_Y) \subset \sigma_f(T)$, but less familiar is the following result for quotient maps:

5.4. **Lemma.** *If $X = Y + Z$ where Y and Z are closed, T-invariant subspaces of X, then $\sigma(T/Z) \subset \sigma_f(T|_Y)$.*

The result follows immediately from the one about restriction operators when X is the *direct* sum of Y and Z, for then the quotient map T/Z is similar to the restriction of T to Y. The general case follows from the restriction theorem and the (easily checked) fact that the map $y + (Y \cap Z) \to y + Z$ is an isomorphism of $Y/(Y \cap Z)$ onto X/Z that establishes a similarity between T/Z and $(T|_Y)/(Y \cap Z)$ (see [14, Proposition 1.2.4] for the details).

With these preliminaries out of the way we can now prove the main result of this section.

5.5. **Theorem.** *If φ is a parabolic linear fractional selfmap of U that is not an automorphism, then C_φ is not supercyclic on H^p for $1 \leq p < \infty$.*

Proof. Recall that $\sigma(C_\varphi)$ is either the closed interval $[0,1]$ or a curve that starts at 1 and converges to the origin by spiralling infinitely often around it, with distance to the origin decreasing monotonically. Choose any numbers $0 < r_1 < \rho_1 < \rho_2 < r_2 < 1$ and note that, because of this monotonicity, $\sigma(C_\varphi)$ intersects $\{|z| > r_1\}$ in an arc that contains the point 1. Let

$$V = \{|z| < \rho_1\} \cup \{|z| > \rho_2\} \quad \text{and} \quad W = \{r_1 < |z| < r_2\},$$

so that $\{V, W\}$ is an open covering of the plane. Because C_φ is decomposable on H^p there exist C_φ-invariant subspaces Y and Z such that $H^p = Y + Z$, $\sigma(C_\varphi|_Y) \subset V$, and $\sigma(C_\varphi|_Z) \subset W$.

Because supercyclicity (indeed any form of cyclicity) is inherited by quotient maps, the proof will be finished if we can show that the quotient map C_φ/Z is not supercyclic on H^p/Z. To this end observe that

$$\sigma(C_\varphi/Z) \subset \sigma_f(C_\varphi|_Y) \subset \sigma_f(C_\varphi) = \sigma(C_\varphi),$$

where the first containment follows from Lemma 5.4, the second was pointed out in §5.3, and the final equality is a consequence of the spiral shape of the H^p-spectrum of C_φ (Corollary 3.10). Thus the spectrum of C_φ/Z lies in V, and therefore decomposes into a disjoint union of two compact sets, $K_1 \subset \{|z| < \rho_1\}$ and $K_2 \subset \{|z| > \rho_2\}$. Lemma 5.1 will then complete the job once we establish that neither K_1 nor K_2 is empty.

For this, recall from §4 that the decomposing subspaces Y and Z were constructed by choosing a C^∞ partition of unity $\{\beta, \gamma\}$ on $\sigma(C_\varphi)$ with $\operatorname{spt}\beta \subset V$ and $\operatorname{spt}\gamma \subset W$, and then setting $Y = E(\operatorname{spt}\beta)$ and $Z = E(\operatorname{spt}\gamma)$. From Lemma 4.1 we know that each point e^{iat} of $\operatorname{spt}\beta$ is an eigenvalue of C_φ for which the corresponding eigenvector e_t lies in Y (here $a \in \Pi_+$ is the translation parameter of φ). Moreover, if $e^{iat} \notin \operatorname{spt}\gamma$ then Lemma 4.1 insures that $e_t \notin Z$, so that the coset $e_t + Z$ is not the zero-element of the quotient space H^p/Z. Thus every point $e^{iat} \in \operatorname{spt}\beta \backslash \operatorname{spt}\gamma$, is a C_φ/Z eigenvalue. Since $\operatorname{spt}\beta \backslash \operatorname{spt}\gamma$ has points in both components of V, and $\sigma(C_\varphi/Z) \subset V$, we see that $\sigma(C_\varphi/Z)$ is split by an origin-centered circle. Thus Lemma 5.1 insures that C_φ/Z is not supercyclic, and therefore neither is C_φ. □

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