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Operators with Dense, Invariant, Cyclic Vector Manifolds*

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We study a class of Banach space operators patterned after the weighted backward shifts on Hilbert space, and show that any non-scalar operator in the commutant of one of these "generalized backward shifts" has a dense, invariant linear manifold whose non-zero members are cyclic vectors. Under appropriate extra hypotheses on the commuting operator, stronger forms of cyclicity are possible, the most extreme being *hypercyclicity* (density of an orbit). Motivated by these results, we examine the cyclic behavior of two seemingly unrelated classes of operators: adjoint multiplications on Hilbert spaces of holomorphic functions, and differential operators on the Fréchet space of entire functions. We show that each of these operators (other than the scalar multiples of the identity) possesses a dense, invariant linear submanifold each of whose non-zero elements is hypercyclic. Finally, we explore some connections with dynamics; many of the hypercyclic operators discussed here are, in at least one of the commonly accepted senses of the word, "chaotic."

INTRODUCTION

A *cyclic vector* for a bounded operator on a Banach space is one whose orbit under that operator has dense linear span. If the orbit itself is dense, the vector is called *hypercyclic*. The importance of cyclic vectors derives from the study of invariant subspaces. The closed linear span of the orbit of a vector is the smallest closed subspace, invariant under the operator, that contains the vector. Thus an operator lacks closed, non-trivial

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invariant subspaces if and only if each non-zero vector is cyclic. Similarly, an operator has no non-trivial closed invariant *subset* if and only if each non-zero vector is hypercyclic.

It is not known if there is a bounded linear operator on separable Hilbert space that does not have closed, non-trivial invariant subspaces. Enflo [14] has constructed a Banach space that supports such an operator. More recently C. Read [27] gave a simpler construction, which he later refined to show that such operators exist on the sequence space l^1 [28]. Read [29] has also constructed a Banach space operator with no proper, closed, non-trivial invariant *subset*. Again, it is not known if such an example can exist on Hilbert space.

The goal of this paper is to unify and extend several different lines of investigation into the properties of cyclic vectors. First there is classical work of G. D. Birkhoff [5] and G. R. MacLane [23] showing that respectively the operators of translation and differentiation, acting on the space of entire functions of one complex variable, have hypercylic vectors. Later Rolewicz [30] showed that any multiple of the backward shift on Hilbert space by a scalar of modulus larger than 1 has hypercyclic vectors (see Section 2). More recently Gethner and Shapiro [16] and Kitai [22] independently found a sufficient condition for hypercyclicity that yields at one stroke the results of Rolewicz, Birkhoff, and MacLane, and provides new examples of operators with hypercyclic vectors.

Then there is recent work of Beauzamy [2–4] which modifies the deep Banach space techniques of Enflo to construct a Hilbert space operator having a dense, invariant linear manifold which, except for zero, consists entirely of hypercyclic vectors (henceforth, we call such a manifold a *hypercyclic vector manifold*, and make similar definitions for other types of cyclicity). The restriction of the operator to that manifold gives an example of a bounded linear operator on a *pre-Hilbert space* with no proper, closed, invariant subset. The guiding hope behind this effort is that the techniques involved might give some insight into how to construct such an example on Hilbert space.

Finally, there is the following result of Hilden and Wallen [20]: *every weighted backward shift on Hilbert space has a supercyclic vector.* A vector is *supercyclic* for an operator if the scalar multiples of the elements in its orbit are dense. Thus hypercyclic implies supercyclic, which in turn implies cyclic.

In the first part of this paper we use elementary sufficient conditions for hypercyclicity, which we present in Section 1, to establish the theorems of both Beauzamy, and of Hilden and Wallen for large classes of operators in very general settings. The first of these classes concerns Banach space operators modelled in "coordinate-free" fashion on the weighted backward shifts for Hilbert space. After a prelimineary discussion of the ordinary

backward shift on Hilbert space, we define these *generalized backward shifts* in Section 3, and prove that every such operator has a dense, invariant supercyclic vector manifold. If the shift is surjective, then every multiple of it by a large enough scalar has a dense invariant *hypercyclic* vector manifold. These results, which generalize the previously mentioned ones of Rolewicz, Beauzamy, and Hilden-Wallen, emerge as special cases of an even more comprehensive theorem:

THEOREM 3.6. (a) *Every bounded operator, not a scalar multiple of the identity, that commutes with a generalized backward shift B has a dense invariant cyclic vector manifold.*

(b) *If the null space of the commuting operator contains that of B, then this manifold can be taken to consist (except for zero) of supercyclic vectors.*

(c) *If, in addition, the commuting operator is surjective, then all sufficiently large scalar multiples of it have dense, invariant hypercyclic vector manifolds.*

This result yields cyclicity theorems for adjoint multiplication operators on certain Hilbert spaces of functions holomorphic on plane domains. However, because of its generality, it does not readily attain the best results in such concrete situations. Thus in Section 4 we abandon backward shifts, and instead apply our original hypercyclicity criteria directly to characterize the hypercyclic adjoint multiplication operators on natural Hilbert spaces of holomorphic functions (Theorem 4.5). We deduce as a corollary that *every non-scalar adjoint multiplication operator is supercyclic.* Once again, we actually obtain dense, invariant manifolds of vectors whose non-zero elements have the desired kind of cyclicity.

In the fifth section of the paper we come the full circle by applying the methods of Sections 1 and 4 to the Fréchet space $H(\mathbb{C}^N)$ of entire functions on $\mathbb{C}^N$, obtaining a result that contains the previously mentioned one-variable theorems of Birkhoff and MacLane:

THEOREM 5.1. *Every continuous linear operator on $H(\mathbb{C}^n)$ that commutes with all translations (or equivalently, commutes with each partial differentiation operator), and is not a scalar multiple of the identity, has a dense, invariant, hypercyclic vector manifold.*

In particular, every linear partial differential operator with constant coefficients and order >0 has a hypercyclic vector.

It should be noted that although our methods produce manifolds of cyclic and hypercyclic vectors far more easily than the methods of Beauzamy; the operators which possess these manifolds are easily seen to possess multitudes of closed invariant subspaces. Thus, unlike the deeper methods of Beauzamy, our work provides no hope for the resolution of the invariant subspace problem for Hilbert space.

Our work suggests a tentative connection between operator theory and dynamics, and we discuss this in the final section. Here we observe that most of the concrete classes of hypercylic operators discussed in the previous sections are actually *chaotic* in the sense of Devaney [12, p. 50], but we give examples to show that some hypercyclic operators do not have this property.

Reader's Guide. The reader interested primarily in function theoretic operator theory might wish to omit Section 3 at first reading (and possibly forever). Then in Section 1 the generality of "hypercyclic sequences of operators" is not needed, and the only sufficient condition required for hypercyclicity is Corollary 1.5.

1. Cyclic Fundamentals

In this section, X will denote a complex Banach space, and T a bounded linear operator on X. T is called a *scalar operator* if it is a scalar multiple of the identity operator on X. Otherwise T is a *non-scalar operator*. Linear subspaces of X will be called *manifolds*. A manifold $\mathcal{M}$ is *invariant* under T if $T\mathcal{M} \subset \mathcal{M}$.

1.1. *Cyclic Vectors.* For $x \in X$, the *orbit* of x under T is the set of images of x under the successive iterates of T:

$$\mathrm{Orb}(T, x) = \{x, Tx, T^2x, \ldots\}.$$

As we mentioned in the Introduction, a vector $x \in X$ is: *hypercyclic* (for T) if $\mathrm{Orb}(T, x)$ is dense in X, *supercyclic* if the set of scalar multiples of the elements of $\mathrm{Orb}(T, x)$ is dense, and *cyclic* if the linear span of $\mathrm{Orb}(T, x)$ is dense. A *hypercyclic operator* is one that has a hypercyclic vector. We similarly define the notion of supercyclic, and cyclic operator.

We call a manifold, each of whose non-zero vectors is cyclic (for T) a *cyclic vector manifold* (for T). We prefer to avoid the terminology "cyclic manifold," which is often used to denote the linear span of an orbit. We define supercyclic, and hypercyclic vector manifolds similarly.

All our results concerning the three kinds of cyclicity defined above will follow from the analysis of a more general notion of hypercyclicity. Suppose $\{T_n : n \in \mathbb{N}\}$ is a sequence of bounded linear operators on X (here $\mathbb{N}$ denotes the positive integers). We say a vector $x \in X$ is *hypercyclic for* $\{T_n\}$ if the collection of images $\{T_n x : n \in \mathbb{N}\}$ is dense in X. If such a vector exists, we call the original sequence of operators hypercyclic. The Baire

Category Theorem provides the following sufficient condition for a sequence of operators to be hypercyclic. Versions of this result have previously appeared in the work of Beauzamy [4, Sect. V, Proposition 2], and Kitai [22, Theorem 2.1].

1.2. Theorem. *Suppose $\{T_n : n \in \mathbb{N}\}$ is a sequence of bounded linear operators on a separable Banach space X. Then the following conditions are equivalent*:

(a) *$\{T_n\}$ has a dense G_δ subset of hypercyclic vectors.*

(b) *For every pair U, V of non-void open subsets of X, there exists a positive integer n such that $T_n(U) \cap V \neq \varnothing$.*

Proof. Fix an enumeration $\{B_n : n \in \mathbb{N}\}$ of the open balls in X with rational radii, and centers in a countable dense subset of X. The continuity of the operators T_n insures that each of the sets

$$G_m = \bigcup \{T_n^{-1}(B_m) : n \in \mathbb{N}\}$$

is open. Now the collection $HC\{T_n\}$ of hypercyclic vectors for $\{T_n\}$ is just

$$HC\{T_n\} = \bigcap \{G_m : m \in \mathbb{N}\},$$

so the hypercyclic vectors form a G_δ set. Condition (b) is equivalent to the assertion that each set G_m is dense (this is purely set theoretic: for any sets E and F, and any mapping T, we have $T(E) \cap F \neq \varnothing \Leftrightarrow E \cap T^{-1}(F) \neq \varnothing$), i.e., that $HC\{T_n\}$ is the intersection of a countable collection of dense open sets. So (b) and the Baire Category Theorem imply (a): $HC\{T_n\}$ itself is dense in X. Conversely, if $HC\{T_n\}$ is dense, then so is each set G_m, hence condition (b) holds. ▌

Remarks. (i) We might call a subset E of X *hypercyclic for* $\{T_n\}$ if the union of the sets $T_n(E)$ $(n \in \mathbb{N})$ is dense in X. Then Theorem 1.2 can be rephrased

> *$\{T_n\}$ has a dense G_δ set of hypercyclic vectors whenever every open set in X is hypercyclic for $\{T_n\}$.*

(ii) In practice it is often easier to use the following equivalent sequential version of condition (b):

(b′) For every pair x, y of points of X there exists a sequence $\{x_k\}$ of points convergent to x, and a subsequence $\{n_k\}$ of positive integers, such that $T_{n_k} x_k \to y$.

(iii) If a *single operator* T is hypercyclic, then it automatically has a dense set of hypercyclic vectors. For if a vector x is hypercyclic for T, then so is $T^n x$ for any positive integer n.

However, the following example, provided by the referee, shows that, in general, a hypercyclic sequence of operators need not have a dense set of hypercyclic vectors. Thus condition (b) of Theorem 1.2 is not necessary for hypercyclicity of a general sequence of operators.

Let H be a two dimensional Hilbert space, with orthonormal basis $\{e, f\}$. Let $\{x_n\}$ be a countable dense subset of H, and for each n, let y_n be a vector of norm n that is orthogonal to x_n. Now define the sequence $\{T_n\}$ of linear operators on H by

$$T_n(\alpha e + \beta f) = \alpha x_n + \beta y_n \qquad (\alpha, \beta \text{ scalars}).$$

Clearly e is a hypercyclic vector for the sequence, as are any of its non-zero scalar multiples. But *there are no others*, since if $\beta \neq 0$, then

$$\|T_n(\alpha e + \beta f)\| \geqslant |\beta| \, \|y_n\| = n\,|\beta| \to \infty \qquad (n \to \infty),$$

hence the set $\{T_n(\alpha e + \beta f)\}$ is not dense in H. So we have a sequence of operators on a two dimensional Hilbert space whose collection of hypercyclic vectors forms a one dimensional subspace.

(iv) It is easy to see that both the collections of cyclic and supercyclic vectors for an operator T are also G_δ sets [7, 36], and that the set of supercyclic vectors is dense as soon as it is non-empty. However, the collection of *cyclic* vectors need not be dense: we will see an example of this phenomenon in Section 3.2.

(v) The following result, due to Kitai [22, Corollary 2.2] follows from the equivalence of Theorem 2.1(b) and hypercyclicity for a single operator:

an invertible operator is hypercyclic if and only if its inverse is hypercyclic.

The point is, once again, that in the statement of (b), the condition $T_n(U) \cap V \neq \emptyset$ can be replaced by $U \cap T_n^{-1}(V) \neq \emptyset$.

The sufficient conditions for hypercyclicity that we will be using for the rest of the paper are special cases of Theorem 1.2.

1.3. Corollary. *A sequence $\{T_n\}$ of bounded linear operators on a Banach space X is hypercyclic if: for each pair U, V of non-void open subsets of X, and each neighborhood W of zero in X, there are infinitely many positive integers n such that both $T_n(U) \cap W$ and $T_n(W) \cap V$ are non-empty.*

Proof. We verify the "sequential version" (b′) of condition (b) of Theorem 1.2 that is indicated in Remark (ii) above. Let x and y be points of X. The hypotheses of the corollary imply that there are sequences (x_k') converging to x, and (x_k'') converging to 0, and a subsequence $\{n_k\}$ of positive integers such that

$$T_{n_k} x_k' \to 0 \qquad \text{and} \qquad T_{n_k} x_k'' \to y.$$

Let $x_k = x_k' + x_k''$. By the linearity of the operators T_n,

$$T_{n_k} x_k = T_{n_k} x_k' + T_{n_k} x_k'' \to 0 + y = y. \quad \blacksquare$$

Remark. Equivalent to the hypothesis of Corollary 1.3 is the apparently weaker requirement that the sets $T_n(U) \cap W$ and $T_n(W) \cap V$ be non-empty for a *single* n.

In practice it is often easier to use the following consequence of Corollary 1.3, where neighborhoods are replaced by operators.

1.4. Corollary. *Suppose $\{T_n\}$ is a sequence of bounded linear operators on X that tends pointwise to zero on a dense subset of X. Suppose further that there is a (possibly different) dense subset Y of X, and a sequence of (possibly non-linear, possibly discontinuous) maps $S_n\colon Y \to Y$ such that $T_n S_n =$ identity on Y for each n, and $\{S_n\}$ tends pointwise to zero on Y. Then $\{T_n\}$ is hypercyclic.*

Proof. Suppose U and V are non-void open subsets of X, and W is a neighborhood of zero in X. The fact that $\{T_n\}$ tends pointwise to zero on a dense subset of X insures that $T_n(U) \cap W$ is non-void for all sufficiently large n. On the other hand, the hypotheses on $\{S_n\}$ guarantee a point $x \in V$ such that $S_n x \to 0$. Therefore, for all sufficiently large n, the vector $S_n x$ belongs to W, hence $x = T_n S_n x \in T_n(W) \cap V$. Thus the hypotheses of Corollary 1.3 are satisfied. $\blacksquare$

In the latter part of this paper we will need only the special case of the above corollary that deals with the sequence of powers of a fixed operator.

1.5. Corollary. *Suppose T is a bounded linear operator on X for which the sequence of powers $\{T^n\}_0^\infty$ tends pointwise to zero on a dense subset of X. If there is a (possibly different) dense subset Y of X, and a map $S\colon Y \to Y$ such that $TS =$ identity on Y, and $\{S^n\}_0^\infty$ tends pointwise to zero on Y, then the operator T is hypercyclic.*

Proof. The hypotheses of Corollary 1.4 are satisfied with $T_n = T^n$, and $S_n = S^n$. $\blacksquare$

Remarks. (a) Corollary 1.4 has stronger hypotheses than Corollary 1.3, and actually gives a stronger conclusion: *every subsequence of* $\{T_n\}$ *is hypercyclic*. A similar conclusion holds for Corollary 1.5, and shows, for example, that under the hypotheses of this corollary, every positive power of T is hyperbolic. It does not seem to be known if every hypercyclic operator has this property [22, Remark 2.13].

(b) We have yet to use the full Banach space structure of X. The definitions of this section could just as well have been formulated for continuous mappings of a metric space. Theorem 1.2 only requires that this metric space be complete, and the proofs of the resulting corollaries go over verbatim to the setting of F-spaces (complete linear metric spaces). We will discuss these matters further in Section 5.

(c) Kitai [22] and Gethner–Shapiro [16] independently proved Corollary 1.5. The fact that the proof given in [16] actually yields Corollary 1.4 was noted by Lech Drewnowski (see [16]).

2. An Example: The Backward Shift on Hilbert Space

As a preview of the ideas and methods that occur in the sequel, we give a quick proof that Rolewicz's original hypercyclic operators actually have dense invariant hypercyclic vector manifolds.

Let H denote a separable Hilbert space, and fix an orthonormal basis $\{e_n : n \geqslant 0\}$ for H. The *backward shift* B on H (relative to the orthonormal basis $\{e_n\}$) is the bounded linear operator B defined on H by

$$Be_n = e_{n-1} \quad \text{if} \quad n \geqslant 1, \quad \text{and} \quad Be_0 = 0. \tag{1}$$

Clearly this definition results in a surjective operator which has norm 1, and therefore cannot by itself be hypercyclic. Here is the main result of this section.

THEOREM. *For each complex number λ of modulus > 1, the operator λB has a dense, invariant hypercyclic vector manifold.*

Proof. Fix $|\lambda| > 1$. Our first task is to use Corollary 1.5 to show that λB has hypercyclic vectors.

Let u denote the "forward shift" operator defined on H by

$$ue_n = e_{n+1} \qquad (n = 0, 1, 2, \ldots).$$

Clearly u is an isometry on H, and

$$Bu = \text{identity on } H.$$

So if $T = \lambda B$, and $S = \lambda^{-1} u$, then $TS = Bu =$ identity on H. As $n \to \infty$,

$$\|S^n\| = |\lambda|^{-n} \to 0.$$

Thus $S^n \to 0$ pointwise on all of H. Now if $x \in \operatorname{span}\{e_n\}$, then $B^n x$ is eventually zero, hence the same is true of $T^n x$. Thus $T^n \to 0$ pointwise on $\operatorname{span}\{e_n\}$, which is a dense subset of H, so all the hypotheses of Corollary 1.5 are satisfied. Thus T is hypercyclic.

We remark that this argument could be replaced by a more elegant topological one that employs Corollary 1.3, along with the fact that B takes the unit ball of H onto itself (see proof of Theorem 3.6(c) for the details). The argument as given serves to preview the proof of part (b) of the Theorem 3.6.

We can now write down a dense invariant hypercyclic vector manifold for $A = \lambda B$. Let $\mathbb{C}[z]$ denote the collection of polynomials in z with complex coefficients (the *holomorphic* polynomials). Let x be a fixed hypercyclic vector for A. We claim that the manifold

$$\mathscr{M} = \{p(B)x : p \in \mathbb{C}[z]\}$$

has the desired properties.

Clearly $\mathscr{M}$ is an invariant manifold for A. It is dense because it contains the orbit of the hypercyclic vector x (for future reference we note that all that is really required for this is that x be cyclic). So it remains to show that every non-zero vector in $\mathscr{M}$ is hypercyclic for A.

To this end, fix p a holomorphic polynomial that is not identically zero. We must show that $p(B)x$ is hypercyclic for A. The key is to show that $p(B)$ has dense range. Once this is done, then we need only observe that

$$\{A^n p(B)x : n \in \mathbb{N}\} = \{p(B)\, A^n x : n \in \mathbb{N}\} = p(B)\{A^n x : n \in \mathbb{N}\}.$$

Since x is hypercyclic for A, $\{A^n x : n \in \mathbb{N}\}$ is dense in H, hence its image under $p(B)$, being the image of a dense set under an operator with dense range, is itself dense. This is what we wanted to show.

For a simple proof that $p(B)$ has dense range, observe that for each $n \geqslant 0$, the matrix of B^n relative to the orthonormal basis $\{e_k\}$ is zero, except for the nth superdiagonal, which consists entirely of one's (the main diagonal is, by definition, the zeroth superdiagonal). So the matrix of a polynomial in B has the kth coefficient of that polynomial on the kth superdiagonal. Thus, for a polynomial p with non-zero constant coefficient, the matrix of $p(B)$ has a non-zero constant on its main diagonal, and therefore maps the linear span Y of the basis vectors $\{e_n\}$ invertibly onto itself.

Any $p \in \mathbb{C}[z]$ can be written $p(z) = z^{\nu} q(z)$, where q has non-vanishing constant coefficient. Thus $p(B)=B^v q(B)$, where $q(B)$ maps Y onto itself. Since B also maps Y onto itself, so does $p(B)$. Since Y is dense in H, the operator $p(B)$ has dense range. This completes the proof that $\mathscr{M}$ is a hypercyclic vector manifold for $A=\lambda B$. ∎

We remark that the theorem shows that B itself has a dense, invariant *supercyclic* vector manifold, since every vector that is hypercyclic for λB is supercyclic for B.

In the sections to follow we will improve Theorem 2.1 considerably, replacing B by a natural Banach space generalization, and λB by an appropriate non-scalar operator that commutes with B. But regardless of how the setting may change, our strategy will always remain the same, namely:

(i) Show that every non-zero polynomial in B has dense range.

(ii) Show that the commuting operator has the desired kind of cyclic vector.

Once these steps have been accomplished, the argument given in the last part of the proof of Theorem 2.1 will apply directly to show that the manifold $\mathscr{M}$ defined there has the desired cyclic properties relative to the commuting operator.

3. Generalized Backward Shifts and their Commutants

Inspired by the backward shift on Hilbert space, we call a bounded linear operator B on a Banach space X a *generalized backward shift* if it obeys the following conditions:

(GBS 1) The kernel of B is one dimensional.

(GBS 2) $\bigcup \{\ker B^n : n=0, 1, 2, ...\}$ is dense in X.

The point is, of course, that if B is a backward shift on Hilbert space, relative to an orthonormal basis $\{e_n\}$, then

$$\ker B^n = \operatorname{span}\{e_0, e_1, e_2, ..., e_{n-1}\}$$

satisfies the two conditions above. More generally, if $\{e_n\}$ is merely an *orthogonal* basis for H for which

$$R := \sup\{\|e_{n-1}\|/\|e_n\| : n=1, 2, ...\} < \infty,$$

then Eqs. (1) of Section 2 still define a bounded linear operator on H, which satisfies axioms (GBS 1) and (GBS 2). Such operators are called

weighted backward shifts on Hilbert space, the terminology reflecting the fact each one arises as the product of an ordinary backward shift and a diagonal "weighting" operator. A. L. Shields' survey article [36] contains much information about the corresponding "forward" versions of these operators.

In order to prove the main result of this section (Theorem 3.6), we need some algebraic results about generalized backward shifts and their commutants. However, before developing these, we note an important class of examples which shows that even on Hilbert space, generalized backward shifts can arise naturally without explicit reference to a basis.

Example (The Bergman Space of Holomorphic Functions). Let Ω be a bounded plane domain, and let $A^2(\Omega)$ denote the *Bergman space* of Ω: the space of holomorphic functions on Ω that are square-integrable with respect to Lebesgue area measure. It is well known that $A^2(\Omega)$ is a closed subspace of the Hilbert space $L^2(\Omega)$, hence is itself a Hilbert space in the L^2 norm. If φ is a bounded holomorphic function on Ω, then M_φ, the operator of "multiplication by φ," defined on $A^2(\Omega)$ by

$$(M_\varphi f)(z)=\varphi(z)f(z) \qquad (f\in A^2(\Omega),\ z\in\Omega)$$

is clearly a bounded linear operator on $A^2(\Omega)$. If $\varphi(z)=z-\alpha$ for some complex number α, then we commit a slight abuse of notation and write $M_{z-\alpha}$ for M_φ. The following result was suggested to us by Sheldon Axler.

3.1. Proposition. *For each point $\alpha\in\Omega$, the Hilbert space adjoint of $M_{z-\alpha}$ is a generalized backward shift.*

Proof. Let $B=(M_{z-\alpha})^*$. Note that

$$\operatorname{Ran} M_{z-\alpha}=\{(z-\alpha)f : f\in A^2(\Omega)\}=\{f\in A^2(\Omega) : f(\alpha)=0\},$$

where the last equality expresses the easily proven fact that if a function $f\in A^2(\Omega)$ vanishes at α, then the holomorphic function $f(z)/(z-\alpha)$ is still square integrable over Ω. Since the functional of evaluation at α is continuous on $A^2(\Omega)$, the equation above shows that the range of $M_{z-\alpha}$ is the kernel of a bounded linear functional, and hence has codimension one. Thus $\ker B=(\text{Range } M_{z-\alpha})^\perp$ has dimension one, so B satisfies condition (GBS 1). To check (GBS 2), observe that if n is a positive integer, then $\ker B^n$ is the orthogonal complement of the range of $(M_{z-\alpha})^n$, which consists of (and actually coincides with all) functions vanishing at α to order $\geq n$. So if $g\in A^2(\Omega)$ is orthogonal to the union of the kernels of the successive powers of B, then it belongs to range of each successive power of $M_{z-\alpha}$, and must, for each n, have a zero of order at least n at α. Thus g

vanishes identically on Ω. This shows that the union of the kernels of the successive powers of B is dense in $A^2(\Omega)$, i.e., B satisfies (GBS 2). ∎

3.2. *Remarks.* (a) We mentioned in Section 1.2, that while the sets of hypercyclic and supercyclic vectors for an operator must be dense whenever they are not empty, this need not be true for *cyclic* vectors. The operator M_z on $A^2 = A^2(U)$ (U = unit disc) furnishes an example: the function 1 is cyclic, since the polynomials are dense in A^2, but the collection of cyclic vectors is not dense. This follows from the fact that norm convergence in A^2 implies uniform convergence on compact subsets of Ω. Thus if a function $f \in A^2$ vanishes at a point of U, so does every member of the closed linear span of $\mathrm{Orb}(M_z, f)$, hence f cannot be cyclic for M_z. So cyclic vectors vanish nowhere on U.

Now suppose f is a member of A^2 that *does* vanish somewhere on U (e.g., $f(z) = z$). By Hurwitz' theorem [1, p. 176] and the connection between norm convergence and uniform convergence on compact subsets of U, no sequence of functions in A^2 without zeros, and in particular no sequence of cyclic vectors, can converge in the norm topology to f. Thus the collection of cyclic vectors for M_z is not dense in A^2.

(b) The argument above shows that no multiplication operator on a non-trivial Banach space of holomorphic functions for which norm convergence implies uniform convergence on compact sets can have a dense set of cyclic vectors.

(c) The argument of Proposition 3.1 works in any Hilbert space H of holomorphic functions with the convergence properties just mentioned in part (b), on which multiplication by z acts as a bounded operator, and which has the *division property*: if $f \in H$ vanishes at a point α of Ω, then the function $f(z)/(z-\alpha)$ still lies in H. For example, Proposition 3.1 remains true for the Hardy space H^2 of the unit disc.

The proof of our main result about cyclicity of operators that commute with generalized backward shifts will require the following algebraic structure theorems. Although these are well known, we present their proofs in order to keep the paper self-contained. The first result, a special case of [26, p. 17, Sublemma 17] shows that every generalized backward shift acts like an "ordinary" backward shift relative to some dense linearly independent set, while the second discusses the matrix representation, relative to this "basis," for operators that commute with a generalized backward shift.

3.3. Proposition (Algebraic Structure of a Generalized Backward Shift). *If a bounded linear operator B on an infinite dimensional Banach space X is a generalized backward shift, then there is a sequence $\{x_n : n \in \mathbb{N}\}$ in X such that*

$$Bx_n = x_{n-1} \quad \text{for each} \quad n > 1, \tag{1}$$

and

$$\ker B = \mathrm{span}\{x_1\}. \tag{2}$$

Any sequence $\{x_n : n \in \mathbb{N}\}$ that satisfies (1) and (2) for a generalized backward shift B also has the following additional properties:

$$\mathrm{Ker}\, B^n = \mathrm{span}\{x_1, x_2, ..., x_n\} \qquad (n \in \mathbb{N}), \tag{3}$$

and

$$\mathrm{span}\{x_n : n \in \mathbb{N}\} \text{ is dense in } X. \tag{4}$$

Proof. For convenience we write $Y_n = \ker B^n$, so in particular $Y_0 = \{0\}$. Clearly, for each non-negative integer n; $Y_n \subset Y_{n+1}$ and $B(Y_n) \subset Y_{n-1}$. We claim more, namely,

$$B(Y_n) = Y_{n-1}, \quad \text{and} \quad \dim Y_n = n \quad (n = 0, 1, 2, ...). \tag{5}$$

The second fact, of course, follows from the first, and the hypothesis that $\dim Y_1 = 1$, but it is convenient to prove both together.

By the definition of generalized backward shift, (5) holds for $n = 1$. Suppose it holds for a certain $n > 1$. Now $Y_n \neq Y_{n+1}$, since otherwise $Y_n = Y_m$ for all $m > n$, contradicting the requirement that the union of all the spaces Y_m should be dense in the infinite dimensional space X. Thus

$$\dim Y_{n+1} \geqslant n + 1. \tag{6}$$

Now consider B as a linear operator from Y_{n+1} to Y_n. We have

$$\begin{aligned} n &= \dim Y_n && [\text{induction hypothesis}] \\ &\geqslant \dim B(Y_{n+1}) && [Y_n \text{ contains } B(Y_{n+1})] \\ &= \dim Y_{n+1} - \dim \ker B && [\text{the "rank plus nullity" theorem}] \\ &= \dim Y_{n+1} - 1. \end{aligned}$$

In view of (6), there is equality thoughout the above display. The last of the resulting equalities asserts that Y_{n+1} has dimension $n+1$, and the second that $B(Y_{n+1})$ has dimension n, so must therefore coincide with Y_n. This induction establishes (5).

Let x_1 be any non-zero member of Y_1. Because of (5) we can inductively choose a sequence $\{x_n : n \in \mathbb{N}\}$ that satisfies (1), i.e., $Bx_n = x_{n-1}$ for all $n > 1$. Recall that (2) is satisfied by the choice of x_1.

Now suppose $\{x_n : n \in \mathbb{N}\}$ is any sequence that satisfies (1) and (2). We need only verify that the sequence also satisfies (3), from which (4) will follow from (GBS 2). Iteration of (1) shows that $B^{n-1}x_n = x_1 \neq 0$, and $B^n x_j = 0$ for all $j \leqslant n$. These statements assert that $x_n \in Y_n \backslash Y_{n-1}$, so the vectors $\{x_n : n \in \mathbb{N}\}$ are linearly independent; and that

$$\{x_1, x_2, ..., x_n\} \subset Y_n.$$

Since Y_n has dimension n, the linearly independent subset $x_1, x_2, ..., x_n$ must span it, hence (3) holds. ∎

For the rest of this section, $\{x_n : n \in \mathbb{N}\}$ is the dense linearly independent set promised for the generalized backward shift B by Proposition 3.3,

$$Y_n = \text{span}\{x_1, x_2, ..., x_n\} = \ker B^n,$$

and

$$Y = \text{span}\{x_n\} = \bigcup \{Y_n : n = 0, 1, 2, ...\}.$$

The last result shows that the matrix for B acting on Y, relative to the basis $\{x_n\}$ is identically one on the first superdiagonal and zero everywhere else. The next result gives the corresponding information about operators that *commute* with B.

3.4. Proposition (Algebraic Structure of the Commutant). *Suppose $A : Y \to Y$ is a (not necessarily bounded) linear transformation that commutes with B. Then each of the subspaces Y_n is A-invariant. Suppose $A \neq 0$. Set*

$$\nu = \min\{n - 1 : n \in \mathbb{N},\ Ax_n \neq 0\}.$$

Then $A = B^\nu A_\nu$, where A_ν: $Y \to Y$ is a linear transformation which commutes with B, and takes each Y_n isomorphically onto itself.

Proof. The kernel of an operator is clearly invariant for anything in the commutant, so each $Y_n = \ker B^n$ is invariant for A.

Suppose $Ax_1 \neq 0$, i.e., $\nu = 0$. Then, because $Y_1 = \text{span}\{x_1\}$ is invariant for A, we have $Ax_1 = \lambda x_1$ for some non-zero scalar λ. Fix a non-negative integer n. Because of the invariance of the subspaces Y_n, we have

$$Ax_n = \sum_{j=1}^{n} a_{jn} x_j$$

for some scalars a_{jn} (i.e., the matrix of A is upper triangular relative to the basis $\{x_n\}$). Upon applying B^{n-1} to both sides of this equation, and using commutativity and the fact that $B^{n-1}x_j = 0$ for $1 \leqslant j < n$, we obtain

$$\lambda x_1 = Ax_1 = AB^{n-1}x_n = B^{n-1}Ax_n = B^{n-1}a_{nn}x_n = a_{nn}x_1.$$

This shows that the upper triangular matrix of A has the constant entry $\lambda \neq 0$ on the main diagonal, so is in particular, an invertible mapping of each subspace Y_n onto itself. This completes the proof in the case $\nu = 0$.

For the general case, we construct the operator A_ν from A and the forward shift operator u: $Y \to Y$ defined by

$$ux_n = x_{n+1} \qquad (n \in \mathbb{N}),$$

and extended linearly to Y. We claim that the linear transformation

$$A_\nu = Au^\nu$$

maps each subspace Y_n onto itself. To see this, first note that A_ν commutes with B on Y (just apply $A_\nu B$ and BA_ν to each of the basis vectors x_n, and observe that the result is the same). Now

$$A_\nu x_1 = Au^\nu x_1 = Ax_{\nu+1} \neq 0 \qquad [\text{definition of } \nu],$$

so by the case discussed above, A_ν is invertible on each Y_n.

The desired factorization follows from the fact that $Bu =$ identity on Y. Indeed

$$B^\nu A_\nu = B^\nu A u^\nu = AB^\nu u^\nu = A. \quad ∎$$

The proof just given shows that the matrix of A relative to the basis $\{x_n\}$ has a constant $\lambda \neq 0$ on the νth superdiagonal, and zeros everywhere below that superdiagonal. Applying this observation to the operator $A - \lambda B^\nu$, and iterating, we could see that *all* the superdiagonals are constant. In other words: every operator on Y that commutes with B can be represented as a formal power series in B. Results like this have been proven for similar classes of operators by Fleming and Jamison [15], and Shields and Wallen [37]. These authors consider the important question of when the formal power series that represents A can be expected to converge to A in any reasonable fashion.

While proving the theorem of Section 2, we needed a special case of Proposition 3.4 to provide an important step in passing from cyclic vectors to cyclic vector manifolds. Proposition 3.4 will play the same role in our work on generalized backward shifts, which is to provide the following corollary.

3.5. COROLLARY. *Every bounded linear operator, other than zero, that commutes with a generalized backward shift, has dense range.*

Proof. Suppose the operator A commutes with the generalized backward shift B. From Proposition 3.4 comes the representation $A = B^\nu A_\nu$, on Y, where A_ν is invertible on Y, and ν is an integer $\geqslant 0$. Thus

$$A(Y) = B^\nu A_\nu(Y) = B^\nu(Y) = Y.$$

The desired result follows from the fact that Y is dense in X. ▮

We are finally ready to prove the main result of this section, which shows that non-scalar operators in the commutant of a generalized backward shift have dense invariant manifolds whose non-zero members exhibit various degrees of cyclicity.

3.6. THEOREM (Cyclic Manifolds for Operators in the Commutant). *Suppose B is a generalized backward shift on a Banach space X, and A is a non-scalar bounded operator that commutes with B. Then*:

(a) *A has a dense, invariant cyclic vector manifold.*

(b) *If* $\ker A \supset \ker B$, *then A has a dense, invariant supercyclic vector manifold.*

(c) *If* $\ker A \supset \ker B$ *and A is surjective, then for all scalars λ of sufficiently large modulus, the operator λA has a dense, invariant hypercyclic vector manifold.*

Proof. We prove the assertions in reverse order.

Proof of (c). Surprisingly, this proof requires none of the algebraic preliminaries of the last few sections. Let X_1 denote the open unit ball of X. We are assuming that A is surjective, so the Open Mapping Theorem provides a positive number ε so that

$$A(X_1) \supset \varepsilon X_1.$$

Thus for every scalar λ of modulus $> 1/\varepsilon$ we have $(\lambda A)^n(X_1) \supset (\lambda\varepsilon)^n X_1$, so because $|\lambda\varepsilon| > 1$,

$$X = \bigcup_{n \geqslant 1} (\lambda A)^n (X_1).$$

The last equation persists if X_1 is replaced on the right side by any ball centered at the origin, and therefore by any neighborhood of zero. In other words, if W is a neighborhood of zero, and V a non-void open subset of X, then

$$(\lambda A)^n (W) \cap V \neq \varnothing \qquad \text{for all sufficiently large } n. \tag{1}$$

Since A commutes with B and its kernel contains that of B, it is easy to check that $\ker A^n \supset \ker B^n$ for each $n \in \mathbb{N}$. In particular, the set

$$\bigcup_{n \geqslant 1} \ker(\lambda A)^n = \bigcup_{n \geqslant 1} \ker A^n$$

is dense in X. Thus if U is a non-void open subset of X, and W a neighborhood of zero, we have

$$(\lambda A)^n (U) \cap W \neq \varnothing \qquad \text{for all sufficiently large } n. \tag{2}$$

Expressions (1) and (2) show that the operator λA satisfies the hypotheses of Corollary 1.3, hence A has hypercyclic vectors.

We produce the required manifold of hypercyclic vectors as in the proof of Theorem 2.1. Fix a hypercyclic vector x for λA, and set

$$\mathcal{M} = \{p(B)x : p \text{ a holomorphic polynomial}\}.$$

As in the proof of Theorem 2.1, $\mathcal{M}$ is a dense submanifold of X. Note that if p is a non-zero holomorphic polynomial, then $p(B)$ commutes with B, and by the matrix representation of B, $p(B)$ is not the zero operator, since its matrix, relative to the basis $\{x_n\}$ has the kth coefficient of p down the kth superdiagonal (counting the main diagonal as the 0th superdiagonal). Thus by Corollary 3.5, $p(B)$ has dense range. The proof that every non-zero element of $\mathcal{M}$ is hypercyclic for λA follows exactly as for corresponding part of the proof of Theorem 2.1, and we omit it. This completes the proof of part (c) of the theorem.

Before proceeding, we note that we could obtain a larger hypercyclic vector manifold for λA by replacing $\mathcal{M}$ by the set of vectors Cx, where C ranges through the entire commutant of A.

In the remaining two cases, we only produce vectors with the required type of cyclicity. The proof that there are dense invariant manifolds whose non-zero elements have the same kind of cyclicity will then follow exactly as above, and need not be mentioned again.

Proof of (b). The fact that $\ker A \supset \ker B$ means that the integer ν of Proposition 3.4 is positive, so we have $A = B^\nu A_\nu$, where maps A_ν each subspace Y_n invertibly onto itself, and its therefore invertible on Y. Define $C\colon Y \to Y$ by

$$C = A_\nu^{-1} u^\nu, \tag{1}$$

where u is forward shift on Y, relative to the basis $\{x_n\}$ defined in the proof of Proposition 3.3. Then C maps Y_n into $Y_{n+\nu}$. Also

$$AC = (B^\nu A_\nu)(A_\nu^{-1} u^\nu) = B^\nu u^\nu = I \qquad (\text{on } Y). \tag{2}$$

Although C need not be bounded on Y, its restriction to the finite dimensional subspace Y_n *is* bounded (every linear operator between finite dimensional Banach spaces is bounded). Let $\sigma(n)$ denote the norm of this restriction. Since these norms form an increasing sequence, it follows that for each vector $x \in Y_k$

$$\begin{aligned}\|C^n x\| &= \|C(C^{n-1}x)\| \\ &\leqslant \sigma(k+(n-1)v)\| C^{n-1}x\| \qquad [\text{since } C^{n-1}(Y_k) \subseteq Y_{k+(n-1)v}] \\ &\vdots \\ &\leqslant \sigma(k+(n-1)v)\,\sigma(k+(n-2)v)\cdots\sigma(k)\,\|x\| \\ &\leqslant [\sigma(k+(n-1)v)]^n\,\|x\| \qquad [\text{since } \sigma(j) \leqslant \sigma(j+1) \text{ for all } j].\end{aligned}$$

Let $r_n = n\sigma(n+(n-1)v)^n$. Set $T_n = r_n A^n$, and on Y define $S_n = r_n^{-1}C^n$. Then by (2) above, $T_n S_n = I$ on Y. Because A maps each Y_n into a preceding one, and Y_1 to $\{0\}$, we see that for each $x \in Y$, the sequence $\{T_n x\}$ is eventually zero. Thus, in order to apply Corollary 1.4 to the sequence $\{T_n\}$, we need only check that $S_n x \to 0$ for each $x \in Y$. But this follows immediately from the definition of r_n and the estimate of the last paragraph: if $x \in Y_k$, then for each $n \geqslant k$,

$$\begin{aligned}\|S_n x\| &= r_n^{-1}\,\|C^n x\| \\ &\leqslant r_n^{-1}[\sigma(k+(n-1)v)]^n\,\|x\| \\ &\leqslant r_n^{-1}[\sigma(n+(n-1)v)]^n\,\|x\| \\ &= (1/n)\,\|x\|,\end{aligned}$$

so, as desired, $\|S_n x\| \to 0$ as $n \to \infty$.

By Corollary 1.4 there is a vector $x \in X$ such that the set $\{T_n x : n \geqslant 1\}$ is dense in X. Since each T_n is a scalar multiple of the original operator A, this vector is supercyclic for A.

Proof of (a). By part (b) we need only consider commuting operators A for which $Ax_1 \neq 0$. From Proposition 3.4 we know that $Ax_1 = \lambda x_1$, where λ is a non-zero scalar. Since A is not a scalar multiple of the identity, the operator $A_0 = A - \lambda I$ is not zero on Y, commutes with B, and annihilates x_1. By part (b), this operator has a (super) cyclic vector x. We claim that x is a cyclic vector for A. This is a standard exercise (cf. [17, Problem 166]: The binomial theorem shows that

$$A_0^n x = (A - \lambda I)^n x \in \operatorname{span}\{x, Ax, ..., A^n x\},$$

so

$$\operatorname{span}\{A_0^n x : n \in N\} \subset \operatorname{span}\{A^n x : n \in N\}.$$

But the set on the left is dense in X, hence so is the one on the right. Thus x is cyclic for A, and the proof is complete. ∎

3.7. *Surjectivity is Necessary in Part* (c). A simple example suffices to show that if the requirement of surjectivity is dropped from the hypotheses of Theorem 3.6(c), then B may not have any scalar multiple with a hypercyclic vector. Let $\{e_k : k \geqslant 0\}$ be an orthonormal basis for a Hilbert space H, and define, for each $k \geqslant 0$, the vector $f_k = (k!)\,e_k$. Let B be the weighted backward shift on H defined relative to the orthogonal basis $\{f_k\}$. Then a straightforward calculation shows that $\|B^n\| = 1/(n!)$, so for each scalar λ,

$$\|(\lambda B)^n\| = \frac{|\lambda|^n}{n!} \to 0 \qquad (n \to \infty),$$

hence λB has no hypercyclic vector.

3.8. *Hypercyclic Backward Shifts.* By contrast with the example above, certain backward shifts are hypercyclic without the help of any scalar multiplication. To see how this can happen, let us assign to each positive valued function β on the non-negative integers, the Hilbert space $H^2(\beta)$ which consists of all formal power series

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n)\,z^n,$$

where

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2\,\beta(n) < \infty.$$

If $\beta \equiv 1$, then $H^2(\beta)$ is the usual Hardy space H^2 of functions holomorphic on the unit disc, while if $\beta(n) = 1/(n+1)$ $(n = 0, 1, 2, ...)$, then $H^2(\beta)$ is the Bergman space of the disc, as discussed in the example at the beginning of this section.

Let B denote the backward shift defined on $H^2(\beta)$ relative to the orthogonal basis $\{z^n\}$. That is, $B(z^{n+1}) = z^n$ if n is a positive integer, and $B(1) = 0$. For the boundedness of B on $H^2(\beta)$ it is necessary and sufficient to have

$$\sup_{n \geqslant 0} \frac{\beta(n+1)}{\beta(n)} < \infty. \qquad (*)$$

The result we are heading for was observed in [16, Sect. 4] by Gethner and Shapiro.

PROPOSITION. *Suppose* (*) *holds. If, in addition,* $\beta(n) \to 0$ *as* $n \to \infty$, *then* B *is hypercyclic on* $H^2(\beta)$.

Proof. The set $\mathscr{P}$ of holomorphic polynomials (i.e., linear combinations of monomials z^n) is clearly dense in $H^2(\beta)$, and for each $p \in \mathscr{P}$ we have $B^n p = 0$ for all sufficiently large n. Let S denote the forward shift, which is defined at least on $\mathscr{P}$ by $S(z^n) = z^{n+1}$ $(n \geqslant 0)$. Then BS is the identity map on $\mathscr{P}$, and the fact that $\beta(n) \to 0$ insures that $S^n \to 0$ pointwise on $\mathscr{P}$. Thus Corollary 1.5 applies, and shows that B is hypercyclic. ▌

3.9. *Reference.* The paper [26] of Nikolskii and Vasunin contains calculations of the multiplicity of the spectrum for a class of operators that contains our generalized backward shifts. In particular, Lemma 7 [26, p. 279] implies that generalized backward shifts are cyclic, and Sub-lemma 17 [26, p. 287] implies our Proposition 3.3. Also related to our work is Theorem 22 [26, p. 293].

3.10. *Prologue to Section* 4. Theorem 3.6 can be interpreted in the context of the Bergman space of a bounded plane domain Ω. For example, if φ is a bounded holomorphic function on Ω then for each $\alpha \in \Omega$, the corresponding multiplication operator M_φ on $A^2(\Omega)$ commutes with $M_{z-\alpha}$, hence its adjoint M_φ^* commutes with the generalized backward shift $M_{z-\alpha}^*$. Since φ is non-constant, M_φ^* is not a scalar multiple of the identity, so by Theorem 3.6(a): *M_φ^* has a dense, invariant cyclic vector manifold.*

As in the proof of Proposition 3.2, if φ vanishes at a point $\alpha \in G$, then

$$\ker M_\varphi^* = (\operatorname{ran} M_\varphi)^\perp = (\varphi A^2(\Omega))^\perp \subset ((z-\alpha)\, A^2(\Omega))^\perp = \ker M_{z-\alpha}^*,$$

so Theorem 3.6(b) implies: *If φ vanishes at some point of Ω, then M_φ^* has a dense, invariant, supercyclic vector manifold.*

Finally, if φ is bounded away from zero in a neighborhood of $\partial\Omega$, then M_φ is bounded below on $A^2(\Omega)$, and hence its adjoint is surjective. If, in addition φ vanishes somewhere on Ω, then Theorem 3.6(c) applies, and shows that: *every multiple of M_φ^* by a scalar of sufficiently large modulus has a dense, invariant, hypercyclic vector manifold.*

In the next section we will use the fact that every adjoint multiplication operator has a rich supply of eigenvectors to obtain a much stronger result: *If φ is a non-constant bounded holomorphic function on Ω, then the operator M_φ^* on $A^2(\Omega)$ has a dense, invariant hypercyclic vector manifold if and only if $\varphi(\Omega)$ intersects the unit circle.* In particular, every M_φ^*, for φ bounded and non-constant, is supercyclic.

We will prove results like these for Hilbert spaces of holomorphic functions on domains in $\mathbb{C}^N$, assuming only minimal hypotheses.

4. HYPERCYCLIC ADJOINT MULTIPLIERS ON HILBERT SPACES OF HOLOMORPHIC FUNCTIONS

In this section Ω denotes a domain (= connected, open set) in $\mathbb{C}^N$, and H is a Hilbert space of functions holomorphic on Ω, subject only to the following two restrictions:

- Non-triviality: $H \neq \{0\}$.
- Bounded point evaluations: For each $z \in \Omega$, the evaluation functional $f \to f(z)$ is continuous on H.

The second hypothesis asserts that convergence in H implies pointwise convergence on Ω. By the boundedness of holomorphic functions on compact sets, along with the uniform boundedness principle, this amounts to requiring that norm convergence imply uniform convergence on compact subsets of Ω. So the restrictions above are satisfied in every naturally occurring situation.

At the end of the last section we used our analysis of generalized backward shifts to provide information about the cyclic behavior of adjoints of multiplication operators on the Bergman space of a bounded plane domain. We now adopt a different method, which leads to much stronger results for the more general spaces H described above. Our main result (Theorem 4.5) requires some preliminary discussion, all of which is well known.

4.1. *Kernel Functions.* For each $z \in \Omega$, the boundedness of point evaluations and the Riesz Representation Theorem provide a unique function $k_z \in H$, called the *reproducing kernel* for z, for which

$$f(z) = \langle f, k_z \rangle \qquad (f \in H).$$

The reader should note that since we are not assuming that the functions in H separate points of Ω, it is possible that different points z of Ω could give rise to the same reproducing kernel. In fact, H could consist only of constant functions, in which case $k_z \equiv 1$ for each $z \in \Omega$.

We will find the following notation convenient. If E is a subset of Ω, then

$$H_E := \operatorname{span}\{k_z : z \in E\}.$$

4.2. PROPOSITION. *If the closure of E contains an open subset of Ω, then H_E is dense in H.*

Proof. If $f \in H$ is orthogonal to k_z for every $z \in E$, then f vanishes identically on E, and hence on its closure. Since this closure contains an open set, and since Ω is connected, $f \equiv 0$ on Ω. Thus $[\operatorname{span}\{k_z : z \in V\}]^\perp = \{0\}$, as desired. ▌

4.3. *Multipliers.* A complex valued function φ on Ω for which the pointwise product $\varphi f \in H$ for every $f \in H$ is called a *multiplier* of H. Each multiplier of H determines a linear *multiplication operator* M_φ by the formula

$$M_\varphi f = \varphi f \qquad (f \in H).$$

The boundedness of point evaluations and the Closed Graph Theorem insure that M_φ is a bounded operator on H. In practice the multipliers of H may be difficult to characterize. Nevertheless, every multiplier must possess the following important properties.

4.4. PROPOSITION. (a) *Every multiplier is a bounded holomorphic function on Ω.*

(b) *If φ is a multiplier, and $z \in \Omega$, then $M_\varphi^* k_z = \overline{\varphi(z)}\, k_z$.*

Proof (Compare [37, p. 782, Lemmas 3 and 4]. (a) We are assuming that H contains a holomorphic function f that does not vanish identically on Ω. Suppose $z \in \Omega$ is not a zero of f. Then for each positive integer n

$$\begin{aligned}|\varphi(z)|^n\, |f(z)| = |M_\varphi^n f(z)| &\leqslant |\langle M_\varphi^n f, k_z \rangle| \leqslant \|M_\varphi^n\|\, \|f\|\, \|k_z\| \\ &\leqslant \|M_\varphi\|^n\, \|f\|\, \|k_z\|.\end{aligned}$$

Upon taking nth roots in the line above, letting $n \to \infty$, and using the fact that $f(z) \neq 0$, we obtain

$$|\varphi(z)| \leqslant \|M_\varphi\|.$$

Thus $|\varphi| \leqslant \|M_\varphi\|$ on $\Omega' = \Omega \setminus \{\text{zeros of } f\}$. Since φ is a multiplier, there exists $g \in H$ such that $\varphi f = g$, so $\varphi = g/f$ is holomorphic on Ω', and bounded on that set by $\|M_\varphi\|$. The multivariable version of Riemann's theorem [32, p. 62, Corollary] asserts that φ has a holomorphic extension to Ω, which is necessarily also bounded by $\|M_\varphi\|$. This completes the proof of part (a). Before proceeding, we note that this proof actually shows that $|\varphi|$ is bounded by the *spectral radius* of M_φ.

(b) For each $f \in H$ and $z \in \Omega$ we have

$$\begin{aligned}\langle M_\varphi^* k_z, f \rangle = \langle k_z, M_\varphi f \rangle = \langle k_z, \varphi f \rangle &= \overline{\varphi(z) f(z)} \\ &= \overline{\varphi(z)} \langle k_z, f \rangle = \langle \overline{\varphi(z)}\, k_z, f \rangle. \quad ▌\end{aligned}$$

We now come to the main result of this section: a sufficient condition for adjoint multipliers on H to be hypercyclic.

4.5. THEOREM. *Suppose φ is a nonconstant multiplier of H. Then M_φ^* is hypercyclic whenever $\varphi(\Omega)$ intersects the unit circle.*

Proof. The existence of a nonconstant multiplier φ guarantees that H is infinite dimensional. For we are assuming that H contains a function f that is not identically zero (although it could be identically constant), and repeated multiplication of this function by φ produces the infinite linearly independent set $\{f, \varphi f, \varphi^2 f, ...\} \subset H$.

Now suppose $\varphi(\Omega)$ intersects the unit circle. Since φ is non-constant, $\varphi(\Omega)$ is open, so the open sets

$$V = \{z \in \Omega : |\varphi(z)| < 1\} \qquad \text{and} \qquad W = \{z \in \Omega : |\varphi(z)| > 1\}$$

are both non-empty. By Proposition 4.2, the linear subspaces H_V and H_W are dense in H. For notational convenience, write $T = M_\varphi^*$. Since

$$T^n = (M_\varphi^n)^* = \text{adjoint of multiplication by } \varphi^n,$$

we have from Proposition 4.4(b)

$$T^n k_z = \overline{\varphi(z)}^n\, k_z \qquad (n = 0, 1, 2, ...).$$

If $z \in V$, so $|\varphi(z)| < 1$, then this yields

$$\|T^n k_z\| \to 0 \qquad (n \to \infty),$$

hence the sequence of operators (T^n) converges pointwise to zero on the dense subset H_V spanned by the kernel functions $\{k_z : z \in V\}$. Thus half the hypotheses of Corollary 1.5 are verified. For the rest, we need to find the "good" right inverse of T.

To see what is involved in this, let us first consider the special case where the collection of reproducing kernels $\{k_z : z \in W\}$ is linearly independent. In this case, we can define a linear map $S : H_W \to H_W$ by extending the definition

$$S k_z = \overline{\varphi(z)}^{-1}\, k_z \qquad (z \in W)$$

linearly to H_W. Since $|\varphi(z)| > 1$ for each $z \in W$, there is no possibility of dividing by zero, and moreover,

$$S^n k_z = \overline{\varphi(z)}^{-n}\, k_z \to 0 \qquad \text{in } H \text{ as } n \to \infty.$$

By definition, $TS = \text{identity}$ on the dense subset H_W of H, so all the hypotheses of Corollary 1.5 are fulfilled, hence $T = M_\varphi^*$ has a hypercyclic vector.

In case the reproducing kernels are not linearly independent, a little more care is required. Enumerate a countable dense subset $W_1 = \{z_n : n \geqslant 1\}$

of W, and inductively choose a subsequence $\{w_n\}$ as follows. Let $z_1 = w_1$. Delete all points $z \in W_1$ for which the kernel function k_z belongs span $\{k_{w_1}\}$. Call the resulting set W_2. Denote the first element of W_2 by w_2. Let W_3 be the set obtained by deleting from W_2 all points z for which k_z belongs span $\{k_{w_1}, k_{w_2}\}$, and let w_3 be the first element of W_3. The infinite dimensionality of H insures that this process never terminates, so continue it indefinitely. The result is an infinite subset $Z = \{w_n\}$ of W for which the corresponding set of kernel functions is linearly independent, and spans a subspace H_Z which coincides with H_{W_1}, and is therefore dense in H by Proposition 4.2. The operator S can now be defined exactly as in the last paragraph, with H_Z in place of H_W. This completes the proof. ∎

Since the image of any non-constant holomorphic function on Ω is an open subset of the complex plane, some multiple of this image must intersect the unit circle. Thus:

4.6. Corollary. *For every non-constant multiplier φ of H, the operator M_φ^* has a hypercyclic scalar multiple, and is therefore supercyclic.*

In the previous sections, we emphasized cyclic vector manifolds. The present situation is no different. The next result shows that the existence of cyclic vectors implies the existence of cyclic vector manifolds.

4.7. Proposition. *If φ is a non-constant multiplier of H, then the operator M_φ^* has dense range.*

Proof. If φ is a non-constant multiplier of H, then M_φ is one-to-one, hence

$$\{0\} = \ker M_\varphi = [\operatorname{ran} M_\varphi^*]^\perp,$$

that is, ran M_φ^* is dense in H. ∎

Using the techniques of the previous sections, the reader can now fill in manifold versions of Theorem 4.5 and Corollary 4.6. For a convenience, we state and prove the first of these.

4.8. Corollary. *Suppose φ is a non-constant multiplier of H and $\varphi(\Omega)$ intersects the unit circle. Then M_φ^* has a dense, invariant hypercyclic vector manifold.*

Proof. Just as in Sections 2 and 3, the result follows from Theorem 4.5 and Proposition 4.7. If f is a hypercyclic vector for M_φ^*, then, as before, the manifold

$$\mathscr{M} = \{p(M_\varphi^*)f : p \text{ a holomorphic polynomial}\}$$

has the desired properties. ∎

The converse of Theorem 4.5 holds in many naturally occurring spaces. Recall from Proposition 4.4 that every multiplier φ of H is a bounded holomorphic function on Ω, with

$$\|\varphi\|_\infty := \sup\{|\varphi(z)| : z \in \Omega\} \leqslant \|M_\varphi\|. \tag{1}$$

For many spaces H, every bounded function is a multiplier, with equality in (1). This is the case, for example, if H is the Bergman space of a bounded domain; or the Hardy space H^2 of either the unit ball [32, Chap. 5] or the unit polydisc [31, Chap. 3]. It is not, however, the case for all spaces. For example, the Dirichlet space $\mathscr{D}$ of the unit disc U, which consists of all functions f holomorphic on U for which the derivative f' is square integrable over U, when taken in the norm

$$\|f\|^2 = |f(0)|^2 + \int_U |f'|^2 \, dA,$$

obeys the hypotheses of this section, but not every bounded function on U is a multiplier. In fact, the characterization of the multipliers of $\mathscr{D}$ is complicated, having been achieved only in the last decade by Stegenga [38].

For more examples, and background on the theory of multipliers on spaces of holomorphic functions on the unit disc, and its connection with weighted shift operators, we refer the reader to Shields' article [36].

4.9. Theorem. *Suppose every bounded function φ on Ω is a multiplier of H, with $\|M_\varphi\| = \|\varphi\|_\infty$. Then for each such φ the operator M_φ^* is hypercyclic if and only if $\varphi(\Omega)$ intersects the unit circle.*

Proof. Since Ω is connected, so is $\varphi(\Omega)$. So if $\varphi(\Omega)$ does not intersect the unit circle, then it lies entirely inside, or entirely outside the unit disc. In the former case,

$$\|M_\varphi^*\| = \|M_\varphi\| = \|\varphi\|_\infty < 1,$$

so M_φ^* cannot be hypercyclic. In the latter case, $1/\varphi$ is holomorphic on Ω, and bounded there by 1, so by the first case, $M_{1/\varphi}^* =$ the inverse of M_φ^* is a contraction on H, hence not hypercyclic. Thus M_φ^* itself is not hypercyclic, by part (b) of the remarks following Theorem 1.2. The converse implication is a special case of Theorem 4.5. ∎

4.10. *Remarks.* (a) The argument just given occurs in more generality in Carol Kitai's thesis [22], where it is the key step in a more comprehensive result (Theorem 2.8): if T is a hypercyclic operator on a Banach space, then every component of the spectrum of T has non-empty intersection with the unit circle.

(b) The idea of using a large supply of eigenvectors to produce cyclic behavior goes back to Clancey and Rogers [11], who make a connection between cyclicity and spectral synthesis, as introduced into operator theory by Wermer [39]. Recently Bourdon and Shapiro [8] employed a related concept to produce *common* cyclic vectors for adjoint multiplications on the spaces considered in this section, thus generalizing earlier work of Wogen [40] and more recent results of K. C. Chan [9]. We do not know if there is a common *supercyclic vector* for the adjoint multipliers on H.

(c) The results of this section also raise the following question: *Does every operator, not a scalar multiple of the identity, that commutes with a generalized backward shift, have a supercyclic vector*? In other words, is the hypothesis "$\ker A \supseteq \ker B$" of Theorem 3.6(b) really needed? Here is a more specific question: *Suppose B is a quasinilpotent backward shift, like the one defined in Section* 3.7. *Is $I+B$ supercyclic*? *Hypercyclic*? We noted in part (a) that the spectrum of every hypercyclic operator must intersect the unit circle. Hence $\lambda I+B$, whose spectrum is the singleton $\{\lambda\}$, cannot be hypercyclic for any scalar λ of modulus $\neq 1$. But perhaps it is always supercyclic.

By contrast, if a generalized backward shift is surjective, then the eigenvalue method used above leads to an improvement of Theorem 3.6(b), at least for the most naturally occurring operators in the commutant.

4.11. THEOREM. *If B is a surjective generalized backward shift on a Banach space X, and F is a non-constant function holomorphic on a neighborhood of the spectrum of B, then the operator $F(B)$ has a dense, invariant supercyclic vector manifold.*

Proof. The point here will be that, quite in contrast with the quasinilpotent case, the spectrum of B will contain a disc of eigenvalues. Since B is surjective, the Open Mapping Theorem provides a positive number r such that for each $y \in X$ there exists $x \in X$ with $Bx = y$ and $\|x\| \leqslant r\,\|y\|$. Thus, starting with

$$x_0 \in \ker B \tag{1}$$

we can choose inductively a sequence $\{x_n\}_0^\infty$ such that for each $n \geqslant 1$,

$$Bx_n = x_{n-1}, \tag{2}$$

and

$$\|x_n\| \leqslant r\,\|x_{n-1}\|. \tag{3}$$

The last inequality shows that

$$\|x_n\| \leqslant r^n\,\|x_0\| \qquad (n = 0, 1, 2, ...),$$

hence for each scalar α of modulus $<r$, the series on the right side of the definition

$$k_\alpha = \sum_{n=0}^{\infty} \alpha^n x_n$$

converges (absolutely) in B, to the vector $k_\alpha \in X$. Moreover, (1) and (2) above show that

$$Bk_\alpha = \alpha k_\alpha \qquad (|\alpha| < r). \tag{4}$$

The proof now proceeds exactly like that of Theorem 4.5. The first order of business is to show that for every open subset V of the disc $rU = \{|\alpha| < r\}$, the linear subspace

$$X_V = \operatorname{span}\{k_\alpha : \alpha \in V\}$$

is dense in X. To see this, suppose Λ is a bounded linear functional on X that annihilates k_α for each $\alpha \in V$. Then the holomorphic function h defined on rU by

$$h(\alpha) = \Lambda(k_\alpha) = \sum_{n=0}^{\infty} \alpha^n \Lambda(x_n) \qquad (\alpha \in rU)$$

vanishes on V, and therefore vanishes identically on rU. Thus $\Lambda(x_n) = 0$ for all n. By (1) and (2) above, and Proposition 3.3, $\operatorname{span}\{x_n\}$ is dense in X, so $\Lambda = 0$ on X. By the Hahn–Banach theorem, the subspace X_V is therefore dense in X.

Now suppose F is a non-constant function holomorphic on the spectrum of B. Then the range of F contains the open set $F(rU)$. Suppose for the moment that $F(rU)$ intersects the unit circle. Then proceeding as before, let

$$V = \{\alpha \in rU : |F(\alpha)| < 1\} \qquad \text{and} \qquad W = \{\alpha \in rU : |F(\alpha)| > 1\}.$$

Since $F(B)\,k_\alpha = F(\alpha)\,k_\alpha$ for each $\alpha \in rU$, we see that the sequence $\{F(B)^n\}$ tends pointwise to zero on the dense subspace X_V, while the right inverse operator S defined on X_W by

$$Sk_\alpha = \frac{1}{F(\alpha)}\,k_\alpha \qquad (\alpha \in W)$$

(the vectors k_α are easily seen to be linearly independent) has the properties required for Corollary 1.5. Thus $F(B)$ has a hypercyclic vector.

It follows quickly from the density of the space X_{rU} that each such holomorphic function of B has dense range, hence the usual argument shows that $F(B)$ has a dense, invariant hypercyclic vector manifold.

Finally, if $F(rU)$ does not intersect the unit circle, then $\lambda F(rU)$ does for some scalar λ. So the corresponding operator $\lambda F(B)$ has a dense, invariant hypercyclic vector manifold, which is the required supercyclic vector manifold for $F(B)$. ∎

5. Hypercyclic Differential Operators

We now apply the methods of Sections 1 and 4 to the Fréchet space $H(\mathbb{C}^N)$ of entire functions on $\mathbb{C}^N$, endowed with the topology of uniform convergence on compact subsets. For $1 \leqslant k \leqslant N$ let D_k denote complex partial differentiation with respect to the kth coordinate, and for $\alpha \in \mathbb{C}^N$, let τ_α denote translation by α

$$\tau_\alpha f(z) = f(z+\alpha) \qquad (f \in H(\mathbb{C}^N),\ z \in \mathbb{C}^N).$$

Both classes of operators are continuous linear transformations taking $H(\mathbb{C}^N)$ into itself.

As we noted in the Introduction, G. D. Birkhoff showed in 1929 that every translation operator is hypercyclic on $H(\mathbb{C})$ [5], and G. R. MacLane obtained the same conclusion in 1952 for the operator of differentiation [23]. These appear to be the first hypercyclicity theorems for linear operators. Thus it seems only fitting to present the following generalization of the theorems of Birkhoff and MacLane.

5.1. Theorem. *Suppose L is a continuous linear operator on $H(\mathbb{C}^N)$ that commutes with each of the translation operators τ_α $(\alpha \in \mathbb{C}^N)$, and is not a scalar multiple of the identity. Then L has a dense, invariant hypercyclic vector manifold.*

Note that this result has no Banach space analogue. For it implies that every scalar multiple of a non-scalar operator that commutes with translations is hypercyclic. But on a Banach space, every multiple of a bounded operator by a sufficiently small scalar is a contraction, and therefore *not* hypercyclic.

Every linear differential operator with constant coefficients commutes with translations. More generally, the operators on $H(\mathbb{C}^N)$ that commute with translations have a well known representation, which we require for the proof of Theorem 5.1, as "infinite order" differential operators.

5.2. Proposition. *For a continuous linear operator L on $H(\mathbb{C}^N)$, the following conditions are equivalent:*

(a) *L commutes with every translation operator τ_α $(\alpha \in \mathbb{C}^N)$.*

(b) *L commutes with each of the differentiation operators D_k $(1 \leqslant k \leqslant N)$.*

(c) *There is a complex Borel measure μ on $\mathbb{C}^N$ with compact support such that*

$$Lf(z) = \int f(z+w)\, d\mu(w) \qquad (z \in \mathbb{C}^N).$$

(d) *$L = \Phi(D)$, where Φ is an entire function on $\mathbb{C}^N$ of exponential type.*

5.3. *Remarks.* Part (d) of Proposition 5.2 requires some explanation. Here $\Phi(D)$ is the operator that results, in the obvious way, from substituting

$$D = (D_1, D_2, ..., D_N) \qquad \text{for} \quad z = (z_1, z_2, ..., z_N)$$

in the power series representation for Φ.

More precisely: to say that Φ is of *exponential type* means that there exist positive constants A and B such that

$$|\Phi(z)| \leqslant A e^{B|z|} \qquad (z \in \mathbb{C}^N).$$

A straightforward computation with power series, and the Cauchy inequalities [21, p. 27] show that this happens if and only if the coefficients in the power series representation

$$\Phi(z) = \sum_{|\nu| \geqslant 0} a_\nu z^\nu$$

obey, for some $R > 0$, the estimate

$$|a_\nu| \leqslant \frac{R^{|\nu|}}{\nu!} \qquad (|\nu| \geqslant 0), \tag{1}$$

(cf. [33, Chap. VII, Sect. 7] for the one variable case), where we adopt the standard notation,

$\nu = (\nu_1, \nu_2, ..., \nu_N)$ is an N-tuple of non-negative integers (a *multi-index*),

$|\nu| := |\nu_1| + |\nu_2| + \cdots + |\nu_N|$ is the "length" of the multi-index ν,

$\nu! := \nu_1!\,\nu_2! \cdots \nu_N!$

$z^\nu := z_1^{\nu_1} z_2^{\nu_2} \cdots z_N^{\nu_N}$, where $z = z(z_1, z_2, ..., z_N) \in \mathbb{C}^N$, and ν is a multi-index.

Moreover, the Cauchy formulas for derivatives [21, p. 27, Formula 2.2.3] show that for each multi-index ν, each $f \in H(\mathbb{C}^N)$, and each $r \geqslant 0$,

$$\|D^\nu f\|_r \leqslant \frac{2^N}{\nu!} \frac{\|f\|_{2r}}{r^\nu}, \tag{2}$$

where

$$\|f\|_r = \sup\{|f(z)| : |z_j| \leqslant r,\ 1 \leqslant j \leqslant N\}. \tag{3}$$

These estimates show that if, for each non-negative integer k we write

$$\Phi_k(z) = \sum_{|\nu| \leqslant k} a_\nu z^\nu,$$

then for every $f \in H(\mathbb{C}^N)$, the sequence

$$\Phi_k(D) f := \sum_{|\nu| \leqslant k} a_\nu D^\nu f \tag{4}$$

converges uniformly on compact subsets of $\mathbb{C}^N$, i.e., $\{\Phi_k(D)\}$ converges pointwise on $H(\mathbb{C}^N)$. We denote the limit operator by $\Phi(D)$. It is easily seen to be continuous on $H(\mathbb{C}^N)$.

As an example of the implication (a) $\Rightarrow$ (d) of Theorem 5.1, the reader might find it amusing to verify that, according to the conventions described above,

$$\tau_\alpha = \exp(\alpha_1 D_1 + \alpha_2 D_2 + \cdots + \alpha_N D_N),$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{C}^N$. This relationship yields the implication (a) $\Rightarrow$ (b) of Proposition 5.2. The converse follows from the easily proven fact that if e_k denotes the kth standard unit vector for $\mathbb{C}^N$, then the operator $(\tau_{\lambda e_k} - I)/\lambda$ tends pointwise to D_k on $H(\mathbb{C}^N)$.

In order to keep the paper self-contained, we will sketch proofs of the other implications of Proposition 5.2. But first we show how this result figures in the proof of Theorem 5.1.

Proof of Theorem 5.1. Suppose L is a continuous linear operator on $H(\mathbb{C}^N)$ that commutes with every translation operator. Then by Proposition 5.2(d) there exists an entire function

$$\Phi(z) = \sum a_\nu z^\nu \qquad (z \in \mathbb{C}^N)$$

on $\mathbb{C}^N$ of exponential type, such that $L = \Phi(D)$, in the sense described above.

In order to use the methods of Section 4, we need a generous supply of eigenfunctions. For each point $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{C}^N$ let

$$e_\alpha(z) = \exp(z_1\alpha_1 + z_2\alpha_2 + \cdots + z_N\alpha_N) \qquad (z \in \mathbb{C}^N).$$

Then for each $1 \leqslant k \leqslant N$,

$$D_k e_\alpha = \alpha_k e_\alpha,$$

so for each multi-index ν,

$$D^\nu e_\alpha = \alpha^\nu e_\alpha,$$

which, along with (4) above shows that

$$L e_\alpha = \Phi(\alpha)\, e_\alpha \qquad (\alpha \in \mathbb{C}^N). \tag{5}$$

We claim that *for each open subset V of $\mathbf{C}^N$, the linear subspace*

$$H_V = \operatorname{span}\{e_\alpha : \alpha \in V\} \tag{6}$$

is dense in $H(\mathbb{C}^N)$. To see this, suppose Λ is a continuous linear functional on $H(\mathbb{C}^N)$ that vanishes on each eigenfunction e_α ($\alpha \in V$). Since the collection of open sets

$$\{f \in H(\mathbb{C}^N) : \|f\|_r < \varepsilon\} \qquad (\varepsilon \text{ and } r > 0)$$

forms a basis for the neighborhoods of zero in $H(\mathbb{C}^N)$, the linear functional L must be bounded in one of the norms $\|\cdot\|_r$. By the Hahn–Banach Theorem and the Riesz Representation Theorem, there is a (non-unique) complex Borel measure μ, supported in the closed ball of radius r, centered at the origin in $\mathbb{C}^N$ such that

$$\Lambda(f) = \int f\, d\mu \qquad (f \in H(\mathbb{C}^N)).$$

In particular,

$$0 = \Lambda(e_\alpha) = \int e_\alpha\, d\mu \qquad (\alpha \in V). \tag{7}$$

Define $F: \mathbb{C}^N \to \mathbb{C}$ by

$$F(\alpha) = \int e_\alpha\, d\mu \qquad (\alpha \in \mathbb{C}^N).$$

Differentiation under the integral sign shows that F is holomorphic on $\mathbb{C}^N$, with

$$D^\nu F(\alpha) = \int z^\nu e_\alpha(z)\, d\mu(z) \qquad (\alpha \in \mathbb{C}), \tag{8}$$

for every multi-index ν. By (7), the entire function F vanishes on the open set V, so it vanishes identically. In particular, all of its derivatives vanish at the origin, so by (8),

$$0 = D^\nu F(0) = \int z^\nu\, d\mu(z) = L(z^\nu) \qquad (\nu \text{ a multi-index}).$$

So L annihilates every monomial, and hence all of $H(\mathbb{C}^N)$. By the Hahn–Banach Theorem, the linear span H_V of the original set of exponentials $\{e_\alpha : \alpha \in V\}$ is dense in $H(\mathbb{C}^N)$, which proves our claim.

Before proceeding further, note that the work of the last paragraph shows that L has dense range whenever it is not the zero operator. For then Φ is not identically zero, so the set of points $\alpha \in \mathbb{C}^N$ at which Φ does not vanish is open and non-empty, hence the set of e_α's corresponding to these points spans a dense subspace of $H(\mathbb{C}^N)$, and each of these eigenfunctions belongs to the range of L.

Now suppose L is not a constant multiple of the identity, so that the entire function Φ is not constant. Thus the open sets

$$V = \{z \in \mathbb{C}^N : |\Phi(z)| < 1\} \qquad \text{and} \qquad W = \{z \in \mathbb{C}^N : |\Phi(z)| > 1\}$$

are both non-empty. For $\alpha \in V$,

$$L^n e_\alpha = \Phi(\alpha)^n e_\alpha \to 0$$

as $n \to \infty$. Thus the sequence of operators $\{L^n\}_0^\infty$ tends pointwise to zero on the dense subspace H_V of $H(\mathbb{C}^N)$.

As in the proof of Theorem 4.5, to obtain the appropriate right inverse S demanded for L by Corollary 1.5, we extend the definition

$$Se_\alpha = \frac{1}{\Phi(\alpha)} e_\alpha \qquad (\alpha \in W)$$

linearly to the dense subspace $H_W = \operatorname{span}\{e_\alpha : \alpha \in W\}$. The resulting map S takes H_W into itself, and LS is the identity map on H_W. Since $|\Phi(\alpha)| > 1$ for each $\alpha \in W$, the sequence $\{S^n\}$ tends pointwise to zero on $\{e_\alpha : \alpha \in W\}$, and therefore on H_W.

The hypotheses of (the Fréchet space version of) Corollary 1.5 are all fulfilled, so L has a hypercyclic vector f. We saw above that every operator that commutes with the translations has dense range, so this is true of every operator $p(L)$, p a holomorphic polynomial. So as in the work of Section 2 and 3,

$$\mathcal{M} = \{p(L)f : p \text{ a holomorphic polynomial}\}$$

is a dense, invariant linear submanifold of $H(\mathbb{C}^N)$ whose non-zero members are all hypercyclic vectors for the operator L. ∎

Remark. As we pointed out above, if L is a continuous linear operator on $H(\mathbb{C}^N)$ that is not identically zero, and commutes with translations, then elementary arguments show L has dense range, and even a densely defined right inverse. In fact much more is true. In the mid-1950's Ehrenpreis [13] and Malgrange [24] independently showed that L is *surjective*, a result recently improved by Meise and Taylor [25] to read: *L has a continuous (everywhere defined) right inverse.*

Proof of Proposition 5.2 (Compare [35, Sect. 23]). The equivalence of (a) and (b) has already been noted.

(a) $\Rightarrow$ (c). We are assuming that L is a continuous linear operator on $H(\mathbb{C}^N)$ that commutes with every translation τ_α. Thus the linear functional, defined on $H(\mathbb{C}^N)$ by

$$\Lambda f = Lf(0) \qquad (f \in H(\mathbb{C}^N)),$$

being the composition of L with the continuous functional of evaluation at the origin, is itself continuous. As in the proof of Theorem 5.1, the Hahn–Banach Theorem and the Riesz Representation Theorem supply a complex Borel measure μ on $\mathbb{C}^N$ with compact support, which represents Λ in the sense

$$\Lambda f = \int f \, d\mu \qquad (f \in H(\mathbb{C}^N)).$$

So for each $z \in \mathbb{C}^N$ and $f \in H(\mathbb{C}^N)$,

$$\begin{aligned}(Lf)(z) &= (\tau_z Lf)(0) = (L\tau_z f)(0) \qquad [L \text{ commutes with } \tau_z] \\ &= \Lambda(\tau_z f) = \int \tau_z f \, d\mu \\ &= \int f(z+w) \, d\mu(w),\end{aligned}$$

which is (c).

(c) $\Rightarrow$ (d). We are given that the measure μ has compact support. Fix $z \in \mathbb{C}^N$. The power series expansion for f, with center at z, converges uniformly on the support of μ, so we can interchange integration and summation in the formula provided by (c),

$$\begin{aligned}Lf(z) &= \int f(z+w) \, d\mu(w) = \int \left(\sum \frac{D^\nu f(z)}{\nu!} w^\nu \right) d\mu(w) \\ &= \sum \frac{\mu_\nu}{\nu!} D^\nu f(z), \end{aligned} \tag{9}$$

where for each multi-index ν,

$$\mu_\nu = \int w^\nu \, d\mu(w).$$

Now for some $R>0$, the support of μ lies in the polydisc $\{|z_k|\leqslant R, 1\leqslant k\leqslant N\}$, so for each multi-index ν,

$$|\mu_\nu|\leqslant R^{|\nu|}\,\|\mu\|.$$

This estimate, along with the inequalities of Section 5.3, shows that the function Φ defined by

$$\Phi(z)=\sum\frac{\mu_\nu}{\nu!}z^\nu \qquad (z\in\mathbb{C}^N)$$

is entire, of exponential type. By the result of calculation (9) above, $L=\Phi(D)$ in the sense described in Remarks 5.3. This establishes (d).

(d) $\Rightarrow$ (a). This follows from the pointwise convergence, on $H(\mathbb{C}^N)$, of the series for $\Phi(D)$, and the chain rule. ∎

5.4. *Two Backward Shifts on* $H(\mathbb{C})$. The definition of generalized backward shift given in Section 3 could just as well have been made for Fréchet spaces. The reader can easily check that on the space $H(\mathbb{C})$ of entire functions of one complex variable, the operator of differentiation is a generalized backward shift. Now parts (a) and (b) of Theorem 3.6 remain true in the context of Fréchet spaces, with just a little more care being needed to prove part (b). However, the proof of part (c) does *not* extend past the Banach setting. According to Theorem 5.1, Theorem 3.6(c) is nevertheless true for operators that commute with differentiation, and this raises the possibility that the result might be true in general for Fréchet spaces.

However, this is not the case: Theorem 3.6(c) is *not* true for the ordinary backward shift B defined on $H(\mathbb{C})$ relative to the monomial basis $\{z^n\}_0^\infty$. That is,

$$Bf(z)=\frac{f(z)-f(0)}{z} \qquad (f\in H(\mathbb{C}),\ z\in\mathbb{C}).$$

PROPOSITION. *No scalar multiple of the backward shift B on $H(\mathbb{C})$ is hypercyclic.*

Proof. Fix $\lambda\in\mathbb{C}$ and $f\in H(\mathbb{C})$. Then for every $r>4|\lambda|$ we have from the Maximum Principle,

$$\|\lambda Bf\|_r:=\max_{|z|\leqslant r}|\lambda Bf(z)|=\max_{|z|=r}|\lambda Bf(z)|\leqslant\frac{2|\lambda|\,\|f\|_r}{r}\leqslant\frac{\|f\|_r}{2}.$$

Upon iterating this inequality n times,

$$\|(\lambda B)^n f\|_r\leqslant 2^{-n}\|f\|_r \qquad (r>4|\lambda|).$$

Thus for every scalar λ, the sequence of powers $\{(\lambda B)^n\}$ converges pointwise to zero on $H(\mathbb{C})$. In particular, the operator λB is not hypercyclic. ∎

5.5. *Remarks.* (a) In addition to his work mentioned in Remarks 4.10 on common cyclic vectors for adjoint multiplications on Hilbert spaces of holomorphic functions, Chan has shown [10] that the collection of linear, constant coefficient partial differential operators of finite (positive) order has a common cyclic vector. Using spectral synthesis methods, Bourdon and Shapiro [8] generalized this result as well to the class of non-scalar operators that commute with translations. Just as for adjoint multiplications, it would be of interest to know if this class of operators, each of which has just been proven hypercyclic, has a *common* hypercyclic vector.

(b) As a consequence of Theorem 5.1 we have the following: *every partial differential operator on* $\mathbb{R}^N$, *not a scalar multiple of the identity, is hypercyclic on* $C^\infty(\mathbb{R}^N)$. To see why this is true, it helps to introduce the restriction operator $\mathscr{R}$ which associates to each entire function $f(z_1, z_2, ..., z_N)$ its real restriction $f(x_1, x_2, ..., x_N)$. $\mathscr{R}$ maps $H(\mathbb{C}^N)$ continuously into $C^\infty(\mathbb{R}^N)$, takes the collection of holomorphic polynomials onto the polynomials in $x_1, x_2, ..., x_N$, and intertwines the corresponding partial differentiation operators,

$$\mathscr{R}\frac{\partial}{\partial z_j}=\frac{\partial}{\partial x_j}\mathscr{R} \qquad (j=1, 2, ..., N).$$

It follows that the same relation holds for polynomials in these operators. From these facts, the reader can easily check that if p is a non-zero holomorphic polynomial, and f a hypercyclic vector for $p(D)$ (acting on the space of entire functions), then $\mathscr{R}f$ is hypercyclic for the operator $p(\partial/\partial x_1, \partial/\partial x_2, ..., \partial/\partial x_N)$, acting on $C^\infty(\mathbb{R}^N)$.

6. Chaotic Linear Operators

In this final section we discuss some aspects of our work that are suggested by dynamics. Devaney [12, p. 50] has proposed that a continuous mapping of a metric space be called *chaotic* if it is topologically transitive (some element has a dense orbit), has a dense set of periodic points, and posseses a certain "sensitivity to initial conditions." Since topological transitivity is, in our setting, hypercyclicity, it makes sense to ask if the operators considered here are actually chaotic. We will show that this is the case for most, but not all, of the examples discussed so far.

We dispense with the issue of sensitive dependence on initial conditions by observing that every hypercyclic operator on an F-space (complete linear metric space) has a dramatic form of this property.

6.1. PROPOSITION. *Suppose X is an F-space, and T a hypercyclic operator on X. Then for every $x \in X$ there is a dense G_δ set of points $S(x) \subset X$, such that the set of orbit-differences $\{T^n x - T^n y : n \geqslant 0\}$ is dense in X for every $y \in S(x)$.*

Proof. According to Remarks (iii) following Theorem 1.2, and (b) following Corollary 1.5, the set $HC(T)$ consisting of all hypercyclic vectors for T is a dense G_δ subset of X, hence so is its translate $S(x) = x + HC(T)$ for each $x \in X$. For every $y \in S(x)$, the vector $y - x$ is hypercyclic for T. The property desired of y follows from this hypercyclicity and the linearity of T. ∎

Observe that if d is the metric on X, and m (possibly $= \infty$) denotes the supremum of distances $d(x, y)$, where x and y run through X, then the proposition implies that corresponding to each point $x \in X$ there is a dense G_δ set of points y for which

$$\limsup_{n \to \infty} d(T^n x, T^n y) = m.$$

Thus in Fréchet spaces, hypercyclicity implies a somewhat stronger version of "sensitive dependence on initial conditions" than is defined in [12, p. 49].

The point is that according to Proposition 6.1, a hypercyclic operator is chaotic if and only if it has a dense set of periodic points. The next result shows that this is true of most of the concrete examples of hypercyclic operators discussed so far.

6.2. THEOREM. *The following linear operators are chaotic:*

(a) *The adjoint multipliers M_φ^* of Section 4, whenever φ is non-constant and $\varphi(\partial\Omega)$ intersects the unit circle.*

(b) *Every continuous linear operator on $H(\mathbb{C}^n)$, not a scalar multiple of the identity, that commutes with every translation (cf. Section 5).*

Proof. By the main results of the sections indicated, each such operator is hypercyclic, so we need only establish the density of their periodic points. For simplicity we concentrate on the one dimensional case, so for example in part (a), Ω is a plane domain.

(a) Suppose H is a space of functions holomorphic on Ω that obeys the hypotheses of Section 4. Let φ be a multiplier of H whose image intersects the unit circle. The domain Ω can be exhausted by an increasing sequence of relatively compact open sets, so we can choose one of these, call it G, so that $\varphi(G)$ intersects the unit circle. Since $\varphi(G)$ is an open subset of the plane, this intersection contains a non-trivial arc of the circle, which in turn contains infinitely many roots of unity. The preimages of these roots of unity form an infinite subset E of G which (since G is relatively compact in Ω) has a limit point in Ω. Just as in the proof of Theorem 4.5, the subspace $H_E = \operatorname{span}\{k_z : z \in E\}$ is therefore dense in H. By Proposition 4.4(b), if $\varphi(z)$ is a root of unity, for example if $z \in E$, then the reproducing kernel k_z is a periodic point for M_φ^*. Since linear combinations of periodic points are again periodic, the dense subspace H_E consists entirely of periodic points. Thus M_φ^* is chaotic, as desired.

(b) The proof here is entirely analogous to the one above. One need only replace the reproducing kernels k_z by the eigenvectors e_α introduced in Section 5. We leave the details to the reader.

In higher complex dimensions the proof is complicated by the fact that the level sets of holomorphic functions are never discrete. Keeping the notation of the proof of part (a), suppose $g \in H$ is orthogonal to the periodic point subspace H_E. Then g vanishes identically on E, and we need to show that this implies $g \equiv 0$. Fix a point $z_0 \in E$, and let L be any complex line through z_0 on which the restriction of φ is non-constant. Then $\varphi(L \cap \Omega)$ is an open subset of the plane that intersects the unit circle, so by the previous argument for part (a) we see that $L \cap \Omega$ has an infinite set of preimages of roots of unity, and this set has an interior limit point. Thus g vanishes identically on $L \cap \Omega$. Since φ is non-constant on $L \cap \Omega$ for all but a finite number of lines L through z_0 (by the Weierstrass Preparation Theorem [32, pp. 290–291], for example), g vanishes identically on a dense subset of Ω, and therefore on Ω itself. Thus, as before, H_E is dense in H. A similar refinement establishes the higher dimensional form of part (b). ∎

Using the same kind of argument, the reader can easily check that the hypercyclic operators occurring in Section 2, Theorem 3.6(c), and more generally in Theorem 4.11 (actually, in its proof), are also chaotic. These results might lead one to wonder if every hypercyclic operator is chaotic. However, this is not the case; it is even possible for a hypercyclic operator to have no periodic points, other than the obvious fixed point at the origin. This is the message of the next result, which concerns the backward shift operator B acting on the space $H^2(\beta)$ introduced in Section 3.8. It shows, for example, that the "Bergman" backward shift, corresponding to weight sequence $\beta(n) = 1/(n+1)$, is hypercyclic, but not chaotic.

Let us recall that the operator B is defined relative to the orthogonal basis $\{z^n\}$ by

$$B(z^n) = z^{n-1} \qquad (n = 1, 2, \ldots), \qquad \text{and} \qquad B(1) = 0;$$

and that in order to insure its boundedness on $H^2(\beta)$ we must require in addition that

$$\sup_{n\geq 0} \frac{\beta(n+1)}{\beta(n)} < \infty. \tag{*}$$

6.3. THEOREM. *Suppose that β satisfies (*) above, and in addition that $\beta(n) \to 0$ as $n \to \infty$. Then the following statements about the action of B on $H^2(\beta)$ are equivalent.*

(a) *B has a periodic point $\neq 0$.*

(b) $\sum_{n=0}^{\infty} \beta(n) < \infty$.

(c) *B is chaotic.*

Proof. (a) $\to$ (b). Suppose B has a non-trivial periodic point f. This means that there exists $f \in H^2(\beta)$, $N > 0$, and $v \geq 0$ such that $B^N f = f$, and $\hat{f}(v) \neq 0$. It follows from the first of these that the coefficient sequence $\{\hat{f}(n) : n \geq 0\}$ is periodic, with period N, and is therefore constant and non-zero on the arithmetic progression $\{v + jN : j \geq 0\}$. Thus

$$\begin{aligned} |\hat{f}(v)|^2 \sum_{j=0}^{\infty} \beta(v+jN) &= \sum_{j=0}^{\infty} |\hat{f}(v+jN)|^2 \beta(v+jN) \\ &\leq \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \beta(n) = \|f\|_2^{\beta} < \infty, \end{aligned}$$

which, since $\hat{f}(v) \neq 0$, implies that

$$\sum_{j=0}^{\infty} \beta(v+jN) < \infty.$$

Now for each $0 \leq k < N$ we can apply the result of the last paragraph above to $B^k f$ instead of f, and deduce that the sequence β is summable over each of the arithmetic progressions $\{v - k + jN : j \geq 0\}$. This proves (b), since these N arithmetic progressions cover the set of non-negative integers.

(b) $\to$ (c). Since $\beta(n) \to 0$, we know from Section 3.8 that B is hypercyclic, and therefore has, in addition, the sensitivity to initial conditions guaranteed by Proposition 6.1. So it remains to show that B has a dense set of periodic points.

The summability of β insures that for each complex number ω of modulus ≤ 1, the power series

$$k_\omega(z) = \sum_{j=0}^{\infty} (\bar{\omega} z)^n$$

belongs to $H^2(\beta)$. Clearly k_ω is an eigenvector for B corresponding to the eigenvalue $\bar{\omega}$, so it is therefore a periodic point of B whenever ω is a root of unity. We claim that

$$V(R) = \operatorname{span}\{k_\omega : \omega \text{ a root of unity}\}$$

is dense in $H^2(\beta)$. The proof is entirely similar to that of Theorem 4.11. Suppose $g \in H^2(\beta)$ is orthogonal to k_ω for each root of unity ω. For each complex number ω of modulus ≤ 1, define

$$F(\omega) = \langle g, k_\omega \rangle = \sum_{j=0}^{\infty} \hat{g}(n)\, \omega^n \beta(n).$$

The summability of β and the Cauchy–Schwarz inequality insure that F is a continuous function on the closed units disc that is holomorphic on the interior. The orthogonality assumption on g means that F vanishes at each root of unity, hence on the entire unit circle. Thus $F(\omega) = 0$ for each ω in the unit disc, so each power series coefficient $\hat{g}(n)\beta(n)$ is zero. Since $\beta > 0$, we conclude that $g = 0$, which proves the claim, and establishes (c).

The implication (c) $\to$ (a) is trivial, so the proof is complete. ∎

6.4. *Closing Remarks.* (a) Domingo Herrero [18] has recently given a spectral characterization of the *closure* of the collection of hypercyclic operators on (separable, infinite dimensional) Hilbert space. One consequence of his result is that the identity operator belongs to the closure of the hypercyclics. Herrero also shows (cf. Theorem 6.3 above) that it is possible for a hypercyclic operator to have only the origin as its periodic point subspace ([18, Proposition 4.7(v)], for example), and he asks if the periodic point subspace of a hypercyclic operator can have finite, non-zero dimension. In [19] Herrero and Zong-yao Wang answer this question by showing that for each non-negative integer n there is a hypercyclic operator with periodic point subspace of dimension n. This result has been obtained independently by the second author (unpublished), who finds the desired operators in the commutant of a quasianalytic bilateral shift.

(b) In [19] Herrero and Wang show that for Hilbert space, every operator in the norm closure of the hypercyclic operators can be written as a perturbation of a hypercyclic operator by an arbitrarily small compact operator. Since the identity operator is in the closure of the hypercyclics (see (a) above) this leads to the somewhat surprising conclusion that some compact perturbation of the identity is hypercyclic.

(c) Examples such as those of Section 4 lead one to believe that perhaps the adjoint of a hypercyclic operator on Hilbert space cannot be hypercyclic. However, Hector Salas [34] has recently shown that this is not the case; *there exist hypercyclic bilateral weighted shifts whose adjoints* *are also hypercyclic.* Necessarily such operators can have no eigenvalues, and therefore can not have non-trivial periodic points. Salas is able to modify his construction to provide yet another proof that for each non-negative integer n there exist hypercyclic operators with periodic point subspace of dimension n.

(d) In [6, 7], Paul Bourdon and the second author study chaotic behavior for composition operators on the Hardy space H^2. This work includes a complete classification of the chaotic composition operators induced by linear fractional transformations of the unit disc into itself, as well as results for more general mappings.

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