1 Introduction

Our setting is a compact metric space $X$ which you can, if you wish, take to be a compact subset of $\mathbb{R}^n$, or even of the complex plane (with the Euclidean metric, of course). Let $C(X)$ denote the space of all continuous functions on $X$ with values in $\mathbb{C}$ (equally well, you can take the values to lie in $\mathbb{R}$). In $C(X)$ we always regard the distance between functions $f$ and $g$ in $C(X)$ to be

$$\text{dist}(f, g) = \max\{|f(x) - g(x)| : x \in X\}.$$

It is easy to check that “dist” is a metric (henceforth: the “max-metric”) on $C(X)$, in which a sequence is convergent if and only if it converges uniformly on $X$. Similarly, a sequence in $C(X)$ is Cauchy iff it is Cauchy uniformly on $X$. Thus the max-metric, which from now on we always assume to be part of the definition of $C(X)$, makes that space complete. These notes prove the fundamental theorem about compactness in $C(X)$:

1.1 The Arzela-Ascoli Theorem If a sequence $\{f_n\}_1^\infty$ in $C(X)$ is bounded and equicontinuous then it has a uniformly convergent subsequence.

In this statement,

(a) “$\mathcal{F} \subset C(X)$ is bounded” means that there exists a positive constant $M < \infty$ such that $|f(x)| \leq M$ for each $x \in X$ and each $f \in \mathcal{F}$, and

(b) “$\mathcal{F} \subset C(X)$ is equicontinuous” means that: for every $\varepsilon > 0$ there exists $\delta > 0$ (which depends only on $\varepsilon$) such that for $x, y \in X$:

$$d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall f \in \mathcal{F},$$

where $d$ is the metric on $X$.

1.2 Exercise. The Arzela-Ascoli Theorem is the key to the following result: A subset $\mathcal{F}$ of $C(X)$ is compact if and only if it is closed, bounded, and equicontinuous.

1.3 Exercise. You can think of $\mathbb{R}^n$ as (real-valued) $C(X)$ where $X$ is a set containing $n$ points, and the metric on $X$ is the discrete metric (the distance between any two different points is 1). The metric thus induced on $\mathbb{R}^n$ is equivalent to, but (unless $n = 1$) not the same as, the Euclidean
one, and a subset of \( \mathbb{R}^n \) is bounded in the usual Euclidean way if and only if it is bounded in this \( C(X) \). Show that every bounded subset of this \( C(X) \) is equicontinuous, thus establishing the Bolzano-Weierstrass theorem as a generalization of the Arzela-Ascoli Theorem.

2 Proof of the Arzela-Ascoli Theorem.

**Step I.** We show that the compact metric space \( X \) is separable, i.e., has a countable dense subset \( S \).

Given a positive integer \( n \) and a point \( x \in X \), let
\[
B(x, 1/n) = \{ y \in X : d(x, y) < 1/n \},
\]
the open ball of radius \( 1/n \), centered at \( x \). For a given \( n \), the collection of all these balls as \( x \) runs through \( X \) is an open cover of \( X \), so (because \( X \) is compact) there is a finite subcollection that also covers \( X \). Let \( S_n \) denote the collection of centers of the balls in this finite subcollection. Thus \( S_n \) is a finite subset of \( X \) that is \( \frac{1}{n} \)-dense in the sense that every point of \( X \) lies within \( 1/n \) of a point of \( S_n \). Clearly the union \( S \) of all the sets \( S_n \) is countable, and dense in \( X \).

**Step II.** We find a subsequence of \( \{f_n\} \) that converges pointwise on \( S \).

This is a standard diagonal argument. Let’s list the (countably many) elements of \( S \) as \( \{x_1, x_2, \ldots\} \). Then the numerical sequence \( \{f_n(x_1)\}_{n=1}^{\infty} \) is bounded, so by Bolzano-Weierstrass it has a convergent subsequence, which we’ll write using double subscripts: \( \{f_{1,n}(x_1)\}_{n=1}^{\infty} \). Now the numerical sequence \( \{f_{1,n}(x_2)\}_{n=1}^{\infty} \) is bounded, so it has a convergent subsequence \( \{f_{2,n}(x_2)\}_{n=1}^{\infty} \). Note that the sequence of functions \( \{f_{2,n} \}_{n=1}^{\infty} \), since it is a subsequence of \( \{f_{1,n} \}_{n=1}^{\infty} \), converges at both \( x_1 \) and \( x_2 \). Proceeding in this fashion we obtain a countable collection of subsequences of our original sequence:

\[
\begin{array}{ccccccc}
& f_{1,1} & f_{1,2} & f_{1,3} & \cdots \\
& f_{2,1} & f_{2,2} & f_{2,3} & \cdots \\
& f_{3,1} & f_{3,2} & f_{3,3} & \cdots \\
& \ddots & \ddots & \ddots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

where the sequence in the \( n \)-th row converges at the points \( x_1, \ldots, x_n \), and each row is a subsequence of the one above it.
Thus the diagonal sequence \( \{f_{n,n}\} \) is a subsequence of the original sequence \( \{f_n\} \) that converges at each point of \( S \).

**Step III. Completion of the proof.**

Let \( \{g_n\} \) be the diagonal subsequence produced in the previous step, convergent at each point of the dense set \( S \). Let \( \varepsilon > 0 \) be given, and choose \( \delta > 0 \) by equicontinuity of the original sequence, so that \( d(x, y) < \delta \) implies \( |g_n(x) - g_n(y)| < \varepsilon/3 \) for each \( x, y \in x \) and each positive integer \( n \). Fix \( M > 1/\delta \) so that the finite subset \( S_M \subset S \) that we produced in Step I is \( \delta \)-dense in \( X \). Since \( \{g_n\} \) converges at each point of \( S_M \), there exists \( N > 0 \) such that

\[
(*) \quad n, m > N \Rightarrow |g_n(s) - g_m(s)| < \varepsilon/3 \quad \forall s \in S_M.
\]

Fix \( x \in X \). Then \( x \) lies within \( \delta \) of some \( s \in S_M \), so if \( n, m > M \):

\[
|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(s)| + |g_n(s) - g_m(s)| + |g_m(s) - g_m(x)|
\]

The first and last terms on the right are \( < \varepsilon/3 \) by our choice of \( \delta \) (which was possible because of the equicontinuity of the original sequence), and the same estimate holds for the middle term by our choice of \( N \) in (*) in summary: given \( \varepsilon > 0 \) we have produced \( N \) so that for each \( x \in X \),

\[
m, n > N \Rightarrow |g_n(x) - g_m(x)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\]

Thus on \( X \) the subsequence \( \{g_n\} \) of \( \{f_n\} \) is uniformly Cauchy, and therefore uniformly convergent. This completes the proof of the Arzela-Ascoli Theorem.