Boundary Value and Eigenvalue Problems

Up to now, we have seen that solutions of second order ordinary differential equations of the form

$$y'' = f(t, y, y') \tag{1}$$

exist under rather general conditions, and are unique if we specify initial values $y(t_0),\ y'(t_0)$. Let us use the notation IVP for the words *initial value problem*.

In many applications, one wants solutions to (1) in which one specifies the *values* of the solution $y(t)$ at two separate points $t_0 < t_1$ rather than specifying the value of $y(t)$ and its derivative at a single point.

This leads to the subject of Boundary Value Problems, a very large and important area of mathematics. The subject is studied for both ordinary and partial differential equations. In the case of partial differential equations, one deals with solutions which are defined on subsets of various Euclidean spaces, and, hence there are many interesting regions for which to specify boundary condtions.

In this course, we will only study two-point boundary value problems for scalar linear second order ordinary differential equations. In most applications, the independent variable of the differential equation represents a spatial condition along a real interval rather than time, so we use x for the independent variable of our functions instead of t.

The general linear second order boundary value problem has the form

$$y'' + p(x)y' + q(x)y = h(x),\ BC \tag{2}$$

Here x is in some interval $I = (a, b) \subset \mathbf{R}$, $p(x), q(x), h(x)$ are continuous real valued functions on I, $\alpha < \beta$ are two fixed real numbers in I, and BC refers to specific boundary condtions.

Let us use the letters BVP to denote *boundary value problem.*

We wish to study all solutions of such a problem. In the cases considered here, we can replace x by $x + \alpha$ if necessary and assume that $\alpha = 0$. We will denote the right boundary point by L.

We will consider four types of boundary conditions, which we denote by the expressions 00, 01, 10, 11. These are defined by

type 00: $\quad y(0) = 0,\ y(L) = 0$

$$\textbf{type 01:} \qquad y(0) = 0,\ y'(L) = 0$$
$$\textbf{type 10:} \qquad y'(0) = 0,\ y(L) = 0$$
$$\textbf{type 11:} \qquad y'(0) = 0,\ y'(L) = 0$$

where $L > 0$.

The BVP

$$y'' + p(x)y' + q(x)y = 0,\ y(0) = 0, y(L) = 0 \tag{3}$$

is called a *homogeneous boundary value problem* and will be denoted by HBVP. Any BVP which is not homogeneous will be called a *non-homogeneous BVP*.

Given a BVP of the form (2) of type 00, 10,01, or 10, there is an associated HBVP of type 00 obtained by replacing $h(x)$ by the zero-function and replacing the boundary conditions by $y(0) = 0,\ y(L) = 0$.

From our experience with IVP's (initial value problems), we might expect that the solutions to a general NBVP are related to those of its associated HBVP. It turns out that BVP's behave very differently than IVP's. For instance, a BVP may have no solution at all, infinitely many solutions, or it may have a unique solution. In a certain sense, BVP's behave more like systems of linear algebraic equations than IVP's.

For comparison, let us recall some general properties of linear algebraic equations.

Consider the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

where A is an $n \times n$ matrix, and $\mathbf{x}$ are $n \times 1$ matrices (which we think of as column vectors). Here, A and $\mathbf{b}$ are known, and we wish to find $\mathbf{x}$.

We have the following facts.

1. if $det(A) \neq 0$, then
   (a) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and
   (b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.
2. if $det(A) = 0$,then
   (a) $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, and

(b) $A\mathbf{x} = \mathbf{b}$ has either no solutions at all or infintely many solutions.

We will restrict our study of BVP's to the case in which p and q are constants and the boundary conditions are of the types 00, 01, 10, and 11.

Let us begin with the HBVP

$$y'' + py' + qy = 0,\ y(0) = 0\ y(L) = 0 \tag{4}$$

Proposition. *If (4) has a non-trivial solution, then $p^2 - 4q < 0$. That is, the characteristic polynomial $z(r) = r^2 + pr + q$ has no real roots.*

Proof. Assume, by way of contradiction that $z(r)$ has real roots.

Case 1: $z(r) = (r - r_1)(r - r_2)$ where $r_1 \neq r_2$.

Then the general solution to the differential equation has the form

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Assume that it is a non-trivial solution, so that either $c_1 \neq 0$ or $c_2 \neq 0$.

The first BC $y(0) = 0$ gives $c_1 + c_2 = 0$, so that $c_1 = -c_2$.

The second BC $y(L) = 0$ gives

$$e^{r_1 L} = e^{r_2 L}$$

But, the function $x \to e^x$ is strictly increasing, so this implies that $r_1 = r_2$, a contradiction.

Case 2: $z(r) = (r - r_1)^2$.

Here the general solution is $y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$.

The first BC, $y(0) = 0$ gives $c_1 = 0$.

Then, the second BC, $y(L) = 0$, gives $c_2 L e^{r_1 L} = 0$. But, since $L > 0$, we have that $L e^{r_1 L} > 0$, so $c_2 = 0$ also. This is a contradiction and the Proposition is proved.

Remark. In a similar manner, one can prove that the BVP's $y'' + py' + qy = 0$ of types 01, 10, and 11 only have non-trivial solutions if $z(r)$ has no real roots. We leave the proof as an exercise.

Thus, in considering constant coefficient BVP's of type 00, 01, 10, 11, we might as well assume that $p^2 - 4q < 0$. The quadratic formula gives that the roots have the form

$$r = -p/2 \pm \sqrt{p^2 - 4q}/2$$

Letting $a = -p/2$ and $b = \sqrt{p^2 - 4q}/2$, we get the general solution to the differential equation as

$$y(x) = e^{ax}\left(c_1 cos(bx) + c_2 sin(bx)\right)$$

It turns out that the methods and ideas in the study of this expression with BC's of types 00, 01, 10, 11 are not much different when $a \neq 0$ or $a = 0$. So, for simplicity, we only consider the case when $a = 0$. That is, the case in which $z(r)$ has purely imaginary roots.

This means that our differential equation has the form

$$y'' + q\ y = 0$$

where $q > 0$. It will simplify things if we write $q = \lambda^2$ where $\lambda > 0$.

Thus, we consider the BVP's of the form

$$y'' + \lambda^2 y = 0,\ \ BC \tag{5}$$

where BC is one of the four types 00, 01, 10, 11.

Remark. We will see below that even for equations of the type (5), non-trivial solutions only occur if there are special relations between λ and L.

Example: An equation of the form (5) with no non-trivial solutions.

Consider

$$y'' + 2y = 0,\ \ y(0) = 0, y(1) = 0\ \ (\text{ so } L = 1).$$

The general solution to the ODE is

$$y(x) = c_1 cos(\sqrt{2}x) + c_2 sin(\sqrt{2}x)$$

The condition $y(0) = 0$, gives $c_1 cos(0) = 0$ or $c_1 = 0$. So, $y(x) = c_2 sin(\sqrt{2}x)$.

The condition $y(1) = 0$ gives $c_2 sin(\sqrt{2}) = 0$, or $\sqrt{2} = n\pi$ for some integer n. But there is no such n.

Conclusion: There are no non-trivial solutions.

Eigenvalue Problems

A real number λ^2 such that the BVP (5) has a non-trivial solution $y_\lambda(x)$ is called an *eigenvalue* of the BVP and the function $y_\lambda(x)$ is called an *eigenfunction* associated to (or corresponding to) λ_n^2. It turns out that if $y_\lambda(x)$ is an eigenfunction, then so is any non-zero multiple $Cy_\lambda(x)$, so we usually just take the constant $C = 1$.

Let us give some examples.

Example 1

Consider the BVP

$$y'' + \lambda^2 y = 0,\ y(0) = 0,\ y(L) = 0 \tag{6}$$

We seek all solutions.

The general solution has the form

$$y(x) = c_1 cos(\lambda x) + c_2 sin(\lambda x)$$

The first BC gives $c_1 = 0$ so we are dealing with the function $c_2 sin(\lambda x)$.

The second BC (if $c_1 \neq 0$) gives that $sin(\lambda L) = 0$.

Thus, $\lambda L = n\pi$ for some positive integer n.

It follows that, for each positive integer n, if

$$\lambda_n = \frac{n\pi}{L}$$

then the function

$$y_n(x) = sin(\frac{n\pi x}{L})$$

is a non-trivial solution of (6).

Thus, the eigenvalues of (6) are the numbers

$$\lambda_n^2 = (\frac{n\pi}{L})^2$$

and the associated eigenfunctions are

$$y_n(x) = sin(\frac{n\pi x}{L})$$

Example 2 Find the eigenvalues and associated eigenfunctions for the BVP

$$y'' + \lambda^2 y = 0, y(0) = 0,\ y'(L) = 0$$

Again, we begin with the general solution.

$$y(x) = c_1 cos(\lambda x) + c_2 sin(\lambda x)$$

BC-1: $y(0) = 0 \Longrightarrow c_1 = 0$.
BC-2: $y'(L) = 0 \Longrightarrow cos(\lambda L) = 0$
The cosine function assumes the value 0 at the odd multiples of $\pi/2$. We are interested in the positive ones, so we can write them as

$$\frac{(2n-1)\pi}{2}, \quad n = 1, 2, 3, 4, \ldots$$

Thus, BC-2 requires that

$$\lambda = \frac{(2n-1)\pi}{2L}$$

for some positive integer n.

Our eigenvalues are

$$\lambda_n^2 = (\frac{(2n-1)\pi}{2L})^2$$

and our associated eigenfunctions are

$$sin(\frac{(2n-1)\pi x}{2L})$$

for $n = 1, 2, 3, \ldots$.

We have found the eigenvalues and associated eigenfunctions for BVP's of types 00 and 01 and the equation

$$y'' + \lambda^2 y = 0$$

Similar techniques work for the types 10 and 01.

We list a type, eigenvalue, eigenfunction table for the equation $y'' + \lambda^2 y = 0$ on the next page.

The results are in the following table. These are all for the differential equation $y'' + \lambda^2 y = 0$.

type	00	01	10	11
eval	$(\frac{n\pi}{L})^2$	$(\frac{(2n-1)\pi}{2L})^2$	$(\frac{(2n-1)\pi}{2L})^2$	$(\frac{n\pi}{L})^2$
efun	$sin(\frac{n\pi x}{L})$	$sin(\frac{(2n-1)\pi x}{2L})$	$cos(\frac{(2n-1)\pi x}{2L})$	$cos(\frac{n\pi x}{L})$