13. Laplace Transform

Review of Improper Integrals

An integral of the form

$$\int_a^b f(t)dt$$

is called an *improper integral* if at least one of the following conditions is satisfied.

1. $a = -\infty$
2. $b = +\infty$
3. $\lim_{t\to a+} f(t) = \pm\infty$
4. $\lim_{t\to b-} f(t) = \pm\infty$

At this time, we only need to consider the case where a is a finite real number and $b = +\infty$. Thus, we consider expressions of the form

$$\int_a^\infty f(t)dt.$$

In this case, we define the value of this expression to be

$$\lim_{b\to\infty} \int_a^b f(t)dt$$

whenever the limit exists. This assumes that the numbers $F(b) = \int_a^b f(t)dt$ exist for each $b > a$, and the limit of the numbers $F(b)$ as b approaches plus infinity exists.

If this limit is finite, then we say the integral *converges*. Otherwise, we say the integral *diverges*.

Let us take some examples.

Example 1:

Fix a positive real number $p > 0$. Let $a > 0$, and consider the improper integral

$$\int_a^\infty \frac{dt}{t^p}.$$

For $a < b$, we have

$$\int_a^b \frac{dt}{t^p} = \frac{t^{-p+1}}{-p+1}\Bigg|_{t=a}^{t=b} = \frac{b^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}$$

We have three cases:

1. $0 < p < 1$.

$$\lim_{b\to\infty} \int_a^b \frac{dt}{t^p} dt = \lim_{b\to\infty} \frac{b^{1-p}}{1-p} - \frac{a^{1-p}}{1-p} = +\infty,$$

   so, the integral diverges.

2. $p = 1$.

$$\lim_{b\to\infty} \int_a^b \frac{dt}{t^p} dt = \lim_{b\to\infty} \int_a^b \frac{dt}{t} = \lim_{b\to\infty} \log(b) - \log(a) = +\infty,$$

   so, the integral diverges again.

3. $p > 1$.

$$\lim_{b\to\infty}\int_a^b \frac{dt}{t^p}dt = \lim_{b\to\infty}\frac{b^{1-p}}{1-p} - \frac{a^{1-p}}{1-p} = -\frac{a^{1-p}}{1-p},$$

so, the integral converges.

Example 2.
Consider the integral

$$\int_0^\infty e^{-ct}dt.$$

where $c > 0$ is a positive real number.
We have

$$\begin{aligned}\int_0^\infty e^{-ct}dt &= \lim_{b\to\infty}\int_0^b e^{-ct}dt \\ &= \lim_{b\to\infty}\frac{-1}{c}e^{-ct}\Big|_{t=0}^{t=b} \\ &= \lim_{b\to\infty}\frac{-1}{c}e^{-cb} + \frac{1}{c} \\ &= \frac{1}{c},\end{aligned}$$

so, the integral always converges.

General Facts about improper integrals:

1. If

$$\lim_{t\to\infty}\frac{f(t)}{g(t)} = L$$

exists and is finite, then

$$\int_a^\infty f(t)dt$$

converges if and only if

$$\int_a^\infty g(t)dt$$

converges.

2. Assume $0 \leq f(t) < g(t)$ for all t. If $\int_0^\infty g(t)dt$ converges, then so does $\int_0^\infty f(t)dt$. On the other hand, if $\int_0^\infty f(t)dt$ diverges, then so does $\int_0^\infty g(t)dt$.

3. If $f(t)$ is a non-negative nonincreasing function on the infinite interval $(0, \infty)$, then

$$\int_1^\infty f(t)dt$$

converges if and only if the infinite series

$$\sum_{n=1}^\infty f(n)$$

converges.

Now, we define the Laplace transform $\mathcal{L}(s)$ of the function $f(t)$ to be the integral

$$\mathcal{L}(s) = \int_0^\infty e^{-st} f(t)dt.$$

This integral may converge only for some values of s, so the Laplace transform of $f(t)$ will only be defined for those values of s.

Let us take some examples.

1. Consider the constant function $f(t) = 1$ for all t. We write its Laplace transform as $\mathcal{L}(1)$. Let's compute it.

$$\begin{aligned}\mathcal{L}(1) &= \int_0^\infty e^{-st} dt \\ &= \frac{-1}{s} e^{-st}\Big|_{t=0}^{t=\infty} \\ &= \frac{1}{s}\end{aligned}$$

   for $s > 0$.

2. Consider the function $f(t) = e^{at}$.

   We have

$$\begin{aligned}\mathcal{L}(e^{at}) &= \int_0^\infty e^{-st} e^{at} dt \\ &= \int_0^\infty e^{-st+at} dt \\ &= \frac{-1}{s-a} e^{-(s-a)t}\Big|_{t=0}^{t=\infty}\end{aligned}$$

$$= \frac{1}{s-a}$$

for $s > a$.

3. For $n \geq 0$, we have

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}. \tag{1}$$

Proof. This is by induction on n.

First we describe the

Principle of Mathematical Induction.

This principle says the following.

Suppose n_0 is an integer, and $S(n)$ represents a statement about integers $n \geq n_0$. If one knows that

(a) $S(n_0)$ is true.

(b) Whenever $S(n)$ is assumed true, it follows that $S(n+1)$ is also true.

Then, one concludes that

$S(n)$ is true for all integers greater than on equal to n_0. (2)

The idea behind this is that if there were some integer $m > n_0$ for which $S(m)$ were false, one could take the least such integer. Let us call it n_1. Then, we know that $S(n_1 - 1)$ is true by the choice of n_1. But, by (b), we know that whever $S(n)$ is true, then so is $S(n+1)$. This would imply that $S(n_1)$ is also true, which contradicts the earlier statement that it was false. Hence, the assumption that there is an $m > n_0$ for which $S(m)$ is false was wrong, and we conclude that (2) is true.

Let us return to the proof of (1).

We have already done the case $n = 0$. Assume it holds for n, we show it holds for $n+1$.

Integrating by parts, and using the inductive assumption for n, we have

$$\begin{aligned}
\mathcal{L}(t^{n+1}) &= \int_0^\infty e^{-st} t^{n+1} dt \\
&= -\frac{1}{s} e^{-st} t^{n+1} \Big|_{t=0}^{t=\infty} - (\int_0^\infty -\frac{1}{s} e^{-st}(n+1)t^n dt \\
&= \frac{1}{s}(n+1)\mathcal{L}(t^n) \\
&= \frac{1}{s}(n+1)\frac{n!}{s^{n+1}}
\end{aligned}$$

$$= \frac{(n+1)!}{s^{n+2}}$$

which is the statement for $n+1$.

4.

$$\mathcal{L}(sin(at)) = \frac{a}{s^2+a^2}$$
$$\mathcal{L}(cos(at)) = \frac{s}{s^2+a^2}$$

These can be proved by integration by parts or using complex variables. We present the second method.

Note that if $f(t) = u(t) + iv(t)$ is a complex valued function of t, then

$$\begin{aligned} \mathcal{L}(f(t)) &= \int_0^\infty e^{-st}(u(t)+iv(t))dt \\ &= \int_0^\infty e^{-st}(u(t))dt + i\int_0^\infty e^{-st}(v(t))dt \\ &= \mathcal{L}(u(t)) + i\mathcal{L}(v(t)) \end{aligned}$$

Using this, we have

$$\begin{aligned} \mathcal{L}(sin(at)) &= \mathcal{L}(Im(e^{iat})) \\ &= Im(\mathcal{L}(e^{iat})) \\ &= Im(\frac{1}{s-ai}) \end{aligned}$$

$$= Im(\frac{s+ai}{s^2+a^2})$$
$$= \frac{a}{s^2+a^2}$$

Also,

$$\mathcal{L}(cos(at)) = \frac{s}{s^2+a^2}.$$

Some useful facts

1. If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{at}f(t)) = F(s-a)$.

   To verify this, notice that

$$\begin{aligned} F(s-a) &= \int_0^\infty e^{-(s-a)t} f(t)dt \\ &= \int_0^\infty e^{-st+at} f(t)dt \\ &= \int_0^\infty e^{-st}e^{at} f(t)dt \\ &= \mathcal{L}(e^{at}f(t)). \end{aligned}$$

2. If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(tf(t)) = -F'(s)$.

   The verification is an application of *differentiation under the integral sign*.

   This states that if $H(s,t)$ and $\frac{dH(s,t)}{ds}$ are continuous functions of t, then $\int_a^b H(s,t)dt$ is a differentiable function of s, and

$$\frac{d}{ds}\left(\int_a^b H(s,t)dt\right) = \int_a^b \frac{dH(s,t)}{ds}dt$$

This holds for finite definite integrals, and, passing to the limit, for convergent integrals of the form

$$\int_a^\infty H(s,t)dt$$

Applying this to $F(s) = \mathcal{L}(f(t))$, we get

$$\begin{aligned}
\frac{dF}{ds} &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt \\
&= \int_0^\infty (\frac{d}{ds}e^{-st}f(t))dt \\
&= \int_0^\infty -te^{-st}f(t))dt \\
&= -\mathcal{L}(tf(t))
\end{aligned}$$

Note that we can iterate this and get, for each $n > 0$,

$$\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$$

where $F^{(n)}(s)$ denotes the $n - th$ derivative of F at s.

General Properties of the Laplace transform.

1. Linearity: The function $\mathcal{L}$ is linear. That is, if $F(s) = \mathcal{L}(f(t))$, and $G(s) = \mathcal{L}(g(t))$ and a, b are constants, then

   $$\mathcal{L}(af(t)+bg(t)) = a\mathcal{L}(f(t))+b\mathcal{L}(g(t)) = aF(s)+bG(s).$$

2. Existence: Let us say that a function f defined on the closed interval $[\alpha, \beta]$ is piecewise continuous if there is a finite set of points $\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_r = \beta$ such that f is continuous on each interval $\alpha_i < t < \alpha_{i+1}$. We say that f is piecewise continuous on the interval $[0, \infty)$ if it is piecewise continuous on each finite subinterval $[0, A]$ for $A > 0$.

   If f is piecewise continuous on the interval $[0, \infty)$ and there are constants $C > 0$ and $a > 0$ such that

   $$| f(t) | \leq Ce^{at}, \quad \forall t \in [0, \infty), \tag{3}$$

   then $\mathcal{L}(f(t))$ exists for $s > a$.

   A piecewise continuous function f satisfying the inequality (3) is said to be of *exponential order.*

3. Uniqueness: if f and g are piecewise continuous and satisfy

   $$| f(t) | \leq Ce^{at}, \text{ and } | g(t) | \leq Ce^{at}, \quad \forall t \in [0, \infty)$$

and $\mathcal{L}(f(t)) = \mathcal{L}(g(t))$ for $s > a$, then $f(t) = g(t)$ except for at most a sequence of points. Indeed for any $A > 0$, there are at most finitely many points in $[0, A]$ at which $f(t)$ and $g(t)$ can fail to be equal.

In the above case, if $\mathcal{L}(f(t)) = F(s)$, we say $\mathcal{L}^{-1}(F(s)) = f(t)$, and we call the operator $\mathcal{L}^{-1}$ the *inverse Laplace transform*. It is defined on those functions which are of exponential order.