Two dimensional systems

We will apply some topology of the Euclidean plane to obtain information about two dimensional planar autonomous systems.

Definition. Let $S^1 = \{z \in \mathbf{C} : | z | = 1\}$ be the unit circle in the plane $\mathbf{R}^2$. A Jordan curve in the plane $\mathbf{R}^2$ (or a simple closed curve) in the plane $\mathbf{R}^2$) is the image of a 1-1 continuous map $h : S^1 \to \mathbf{R}^2$.

Theorem (Jordan Curve Theorem). *Let γ be a Jordan curve in the plane $\mathbf{R}^2$. Then, $\mathbf{R}^2 \setminus \gamma$ is the union of two disjoint open connected sets S_1, S_2 each of which have γ as boundary. Precisely one of the regions S_1, S_2 is bounded.*

Remark. The book refers to the sets S_i as being arcwise connected. But open connected sets in the plane are arcwise connected.

We will not prove this theorem here, instead referring to a course in topology. The bounded region of $\mathbf{R}^2 \setminus \gamma$ is frequently referred to as the *interior* of γ although it is *not* the interior in the sense of topology.

Unless otherwise stated, we will assume that f is a C^1 vector field defined in the plane $\mathbf{R}^2$ and, for each x, the solution $\phi(t, x)$ is defined for all $t \in \mathbf{R}$.

Let p be a regular point of the vector field f; i.e., $f(p) \neq 0$. Let L be a closed tranversal to f at p. This means that there is a C^1 diffeomorphism $h : [-1, 1] \to L$ such that $h(0) = p, h'(t) \neq 0$, and, for each $t \in [-1, 1], h'(t)$ is not a multiple of $f(h(t))$. Let L_0 be the interior of L; i.e., $L_0 = \{h(s) : -1 < s < 1\}$.

Let $V = \{q \in L_0 : \text{there is a } t_q > 0 \text{ with } \phi(t_q, q) \in L_0 \text{ and } \phi(t, q) \in \mathbf{R}^2 \setminus L_0 \text{ for } 0 < t < t_q\}$. The set V is the set of points in L_0 whose positive orbits return to L_0. Let $V' = \{\phi(t_q, q) : q \in V\}$. Set $W = h^{-1}V, W' = h^{-1}V'$ so that W and W' are subsets of $(-1, 1)$.

Define $g : W \to W'$ by $g(w) = h^{-1}\phi(t_{h(w)}, h(w))$.

See Figure 1

Lemma. *The set W is open in $(-1, 1)$ and the function g is continuous on W. For any $z \in W$ for which the iterates $z, g(z), \ldots, g^n(z)$ is defined, the sequence $z, g(z), \ldots, g^n(z)$ is monotone in $(-1, 1)$.*

Proof.

W is open:

Fix $w \in W$, we want to see that if z is near w in $(-1, 1)$, then $z \in W$ also. We can find a flow-box B centered at $\phi(t_{h(w)}, h(w))$ such that each connected component of an orbit in B is an arc which only meets L in one point. Moreover, there is an $\epsilon_1 > 0$ such that if $u \in B$, then there is a $\eta(u) \in$

$(-\epsilon_1, \epsilon_1)$ such that $\phi(\eta(u), u) \in L$. Moreover, in the flow-box coordinates on B, $\eta(u)$ is obviously a continuous function of u since $L \cap B$ is the graph of a continuous function from the vertical direction to the horizontal direction. This implies that, in the standard coordinates on B, $\eta(u)$ is still a continuous function of u.

Now, let $s_w = t_{h(w)}$. Then, for z near w in W, $\phi(s_w, h(z))$ is near $\phi(s_w, h(w))$ so it is in B (since $\phi(t, x)$ is a continous function of (t, x)). Then,

$$\phi(\eta(\phi(s_w, h(z))), \phi(s_w, h(z))) = \phi(\eta(\phi(s_w, h(z))) + s_w, h(z)) \in L.$$

This gives that $h(z) \in W$, so W is open. Also, since

$$g(z) = h^{-1}\phi(\eta(\phi(s_w, h(z))) + s_w, h(z)),$$

we get that g is continuous.

We now prove the monotonicity statement.

Consider a point $z \in W$. If $g(z) = z$ for all z, there is nothing to prove, so assume z is such that $g(z) > z$. In the opposite case in which $g(z) < z$ one proceeds similarly.

The solution curve from $h(z)$ to $h(g(z))$ together with the piece, say L_1 of L from $h(g(z))$ to $h(z)$ is a Jordan curve γ. Since solutions always cross L moving in the same direction, the forward orbit of a point in the interval $L_1 \setminus \{h(z)\}$ always lies in the same component of the complement of γ. If $g(g(z))$ is defined, then $g(g(z))$ must be greater than $g(z)$ since otherwise, the forward orbit of $h(g(z))$ would have to pass from one component of the complement of γ to the other one. Now the argument continues replacing z by $g(z)$. QED.

Corollary. *The $\omega-$limit set $\omega(\gamma)$ of an orbit γ can intersect the interior L_0 of a transversal L in at most one point. If $\gamma = \gamma(p)$ is the orbit through p, and $\omega(\gamma)$ meets the interior of a transversal at p_0, then, either $\omega(\gamma) = \gamma$ in which case γ is periodic or the points in the forward orbit $O_+(p)$ in L_0 approach p_0 monotonically in L_0.*

Proof.

We assume that L_0 is small enough that it fits inside a single flow box B.

Since $\omega(p) \cap L_0 \supset \{p_0\}$, there is a sequence $t_1 < t_2 < \ldots$ with $t_k \to \infty$ such that $\phi(t_k, p) \to p_0$ in L_0.

Case 1:

For some $j < k$, $\phi(t_j, p) = \phi(t_k, p)$. Then, γ is periodic. It must be equal to its own $\omega-$limit set.

Case 2:

For all $k < j$, $\phi(t_j, p) \neq \phi(t_k, p)$.

Constructing a Jordan curve using pieces of orbits $\phi(t_i, p), \phi(t_{i+1}, p)$ and pieces of L_0 as before shows that the forward orbit $O_+(p)$ in L_0 approach p_0 monotonically in L_0 as required. QED.

Corollary. *If some regular point of $O_+(p)$ is also in $\omega(p)$, then $O(p)$ is periodic.*

Proof. This is another corollary of the Jordan curve theorem and the flow box theorem.

Theorem. *A bounded minimal set of a C^1 autonomous planar vector field is a critical point or a periodic orbit.*

Proof.

If the minimal set is not a critical point, then it contains no critical points. But each of its orbits must be dense in the set. Thus, each of its points is in its own $\omega-$limit sets. Hence, by the previous corollary, each of its orbits is periodic. Since it is minimal, it must be a single orbit. QED.

Theorem (Poincare-Bendixson). *Suppose that $O_+(x)$ is a bounded positive semi-orbit of an autonomous C^1 vector field f in the plane. If $\omega(x)$ does not contain a critical point, then $\omega(x)$ consists of a periodic orbit $O(p)$. Either $O(p) = O(x)$ or $O(p) = Closure(O_+(x)) \setminus O_+(x)$.*

Proof.

Since $O_+(x)$ is bounded, $Closure(O_+(x)) \neq \emptyset$ and $\omega(x)$ is a non-empty, invariant set. By hypothesis, it contains only regular points. It also contains a minimal set Σ which must be a periodic orbit, $O(p)$. Let L_0 be a small open transversal arc to the periodic orbit $O(p)$ at p. Since, p is in $\omega(x)$, there is a sequence $t_1 < t_2 \to \infty$ such that $\phi(t_i, x) \in L_0$ and $\phi(t_i, x) \to p$. as $i \to \infty$. Let $z_i = \phi(t_i, x)$. If, for some i_0, $z_i = z_{i+1}$, then $O(x)$ is periodic and must equal $O(p)$. If not, then the sequence of points z_i with different i is a sequence of distinct points in L_0. By the flow-box theorem and the Jordan curve theorem, the sequence z_i converges monotonically to p. It follows that $O(p) = Closure(O_+(x)) \setminus O_+(x)$. QED

Lemma. *Suppose that $\omega(p_0)$ contains a regular point, p_1 which is not in the orbit of p_0. Then, p_0 cannot be in $\omega(x)$ for any x.*

Proof.

Take a small open transversal L_0 to p_1. The positive orbit of p_0 must cross L_0 and monotonically converge to p_1. Then, pieces of this orbit together with pieces of L_0 form Jordan curves which trap the positive orbit of any point near p_0. Hence, p_0 cannot be in $\omega(x)$ for any x. QED.

Theorem. *Suppose $O_+(x)$ is a positive semi-orbit in a closed bounded subset K of the plane for a C^1 vector field f. Assume that K contains only a finite number of critical points. Then, one of the following holds.*

(i) *$\omega(x)$ is a critical point.*

(ii) *$\omega(x)$ is a periodic orbit.*

(iii) *$\omega(x)$ consists of a finite number of critical points and a set of orbits γ_i such that each γ_i has its $\omega-$limit set $\omega(\gamma_i)$ and $\alpha-$limit set $\alpha(\gamma_i)$ consisting of a critical point.*

Definition. A *cycle* of critical points is a finite sequence $p_1, p_2, \ldots, p_n$ of critical points such that $p_1 = p_n$ and, for each $1 \leq i < n$, there is a point x_i such that $\alpha(x_i) = p_i$ and $\omega(x_i) = p_{i+1}$. A solution γ whose α and ω limit sets are critical points is called a *separatrice.*

Remark. One way in which condition (iii) occurs is that $\omega(x)$ consists of separatrices of a cycle of critical points. There can also be several regular orbits whose α and ω limits are the same critical point.

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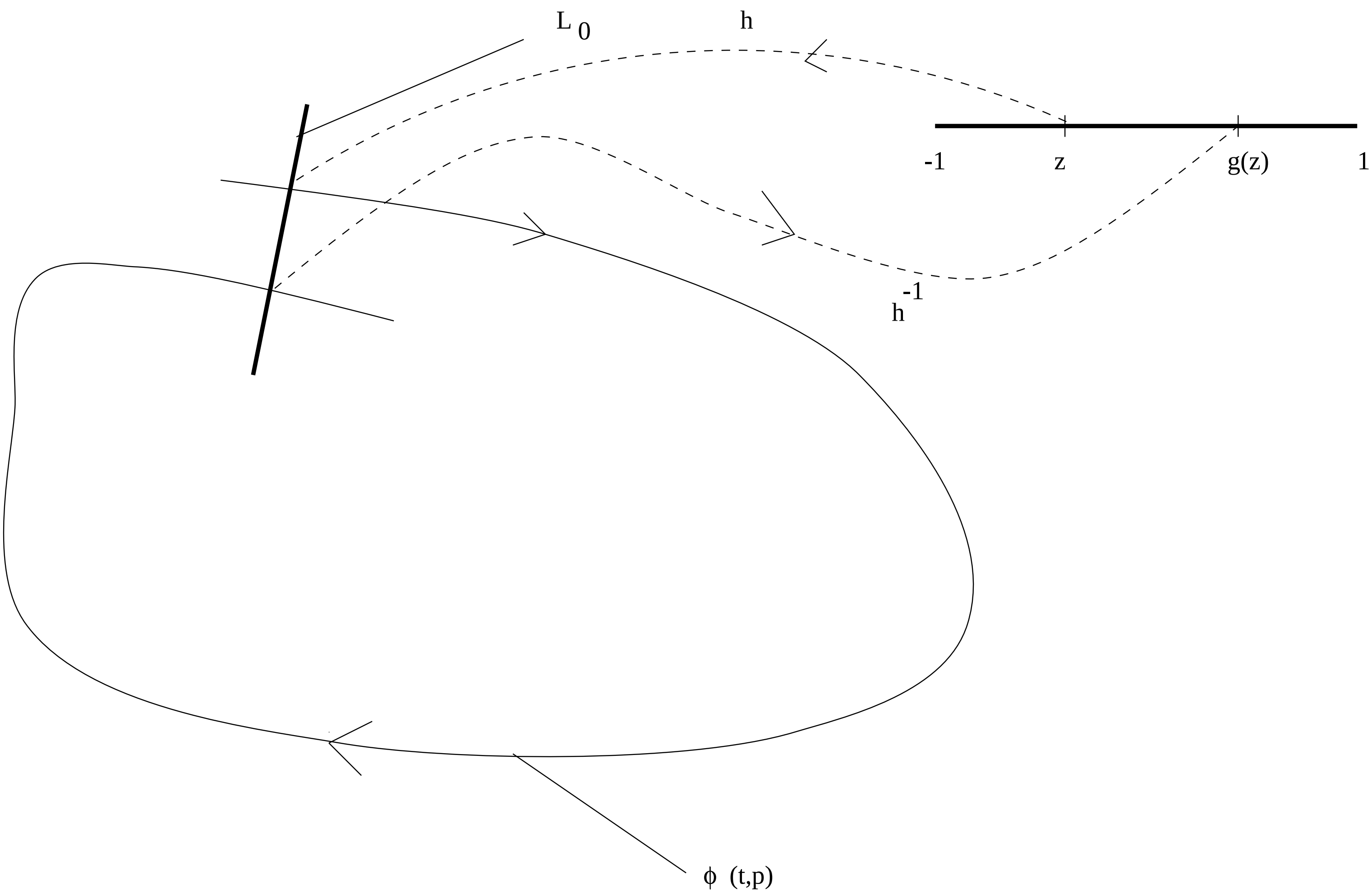
Proof.

If $\omega(x)$ contains no regular points, then since it is connected, it must consist of a single critical point. This is case (i).

Thus, we may assume $\omega(x)$ contains at least one regular point, say p_0. If $O(p_0)$ is periodic, and L_0 is an open transversal at p_0, then $O_+(x) \bigcap L_0$ converges monotonically to p_0, so $\omega(x) = O(p_0)$ which is case (ii).

If the orbit $O(p_0)$ is not periodic, then its $\omega-$limit set must be disjoint from $O(p_0)$. If $\omega(p_0)$ contained a regular point, then the previous lemma would contradict the assumption that p_0 is itself an $\omega-$limit point (of x). Therefore, $\omega(p_0)$ consists only of critical points. Since it is connected, it must be a single critical point. A similar argument works for $\alpha(p_0)$.

We have therefore proved that each regular, non-periodic, $\omega-$limit point or $\alpha-$limit point of x must have each of its own $\omega-$limit and $\alpha-$limit sets reducing to single critical points. QED.



L 0
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