Index for planar $C^2$ vector fields

Let $\Omega$ be an open set in $\mathbb{R}^n$. We say that $\Omega$ is connected if it cannot be written as the disjoint union of two non-empty open subsets.

A curve or path in $\Omega$ is a continuous map $\gamma : [a, b] \to \Omega$ where $[a, b]$ is a closed interval in $\mathbb{R}$. Sometimes we abuse the language by referring to the image $\text{Image}(\gamma)$ for such a $\gamma$ as a curve or path. We can always reparametrize a curve $\gamma$ so that it is defined on the closed unit real interval $[0, 1]$ and its image is unchanged. We simply use $\gamma_1(t) = \gamma((1 - t) * a + t * b)$.

We say that $\Omega$ is path connected if for any two points $p_1, p_2 \in \Omega$, there is a curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = p_1$ and $\gamma(1) = p_1$.

It is a simple exercise to verify that an open connected subset of $\mathbb{R}^n$ is path connected.

An region in $\mathbb{R}^n$ is defined to be an open connected set.

Let $\gamma_i : [0, 1] \to \Omega$, $i = 1, 2$ be two curves in the region $\Omega \subset \mathbb{R}^n$.

We say that $\gamma_1$ is homotopic to $\gamma_2$ if there is a continuous map $F : [0, 1] \times [0, 1] \to \Omega$ such that

$$F(t, 0) = \gamma_1(t) \text{ and } F(t, 1) = \gamma_2(t)$$
for all $t \in [0, 1]$. This is the precise way of saying the
$\gamma_1$ can be \textit{continuously deformed} into $\gamma_2$.

We call $F$ a \textit{homotopy} from $\gamma_1$ to $\gamma_2$.

When this is the case, we write $\gamma_1 \simeq \gamma_2$.

Given a curve $\gamma : [0, 1] \to \Omega$, define its \textit{negative} or
\textit{inverse} or \textit{reverse} curve $-\gamma(t)$ by

$$\gamma(t) = \gamma(1 - t).$$

If $\gamma_1, \gamma_2$ are $C^r$ for $r \geq 1$, then we say that $\gamma_1$ is $C^r$
homotopic to $\gamma_2$ if the map $F$ above can be taken to be $C^r$.

\textbf{Facts.}

1. The relation $\simeq$ is an equivalence relation on curves
   (parametrized by $[0, 1]$).

2. If $\gamma_1 \simeq \gamma_2$, then $-\gamma_1 \simeq -\gamma_2$.

If $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ and a homotopy
$F$ can be chosen so that $F(0, s) = \gamma_1(0)$ and $F(1, s) = \gamma_1(0)$ for all $s$, then we say that $\gamma_1$ is homotopic to $\gamma_2$
\textit{relative to their boundaries} and we write $\gamma_1 \simeq_\partial \gamma_2$.

A curve $\gamma : [0, 1] \to \Omega$ is called a \textit{loop} or \textit{closed curve}
in $\Omega$ if $\gamma(0) = \gamma(1)$.

A \textit{constant curve} in $\Omega$ is a continuous map $\gamma : [0, 1] \to \Omega$
such that $\gamma(t) = \gamma(s)$ for all $s \in [0, 1]$. Thus, the image is a single point.
A loop $\gamma$ in $\Omega$ is *null-homotopic* or *in-essential* if it is homotopic to a constant curve. Sometimes, we say it is *homotopic to a constant* or *homotopic to a point*.

A region $\Omega$ is called *simply connected* if every loop in $\Omega$ is null-homotopic.

**Examples**

1. An open ball is simply connected.

2. The complement of a point in $\mathbb{R}^2$ is not simply connected.

3. The unit circle in $\mathbb{R}^2$ is not simply connected.

4. If $n \geq 3$, then the complement of any finite (non-empty) set in $\mathbb{R}^n$ is simply connected. Also, any sphere $S^n$ (i.e., boundary of a ball) is simply connected.

5. The two dimensional torus $S^1 \times S^1$ is not simply connected.

Let $(x, y)$ represent the coordinates of a point in $\Omega$, and let $P(x, y), Q(x, y)$ be $C^1$ real-valued functions from $\Omega$ to $\mathbb{R}$.

The formal expression

$$\omega(x, y) = P(x, y)dx + Q(x, y)dy$$
is called a \((C^1)\) differential 1-form (or simply a 1-form) \(\Omega\). We sometimes write \(\omega = Pdx + Qdy\) leaving out the explicit dependence on \((x, y)\).

Given a \(C^1\) curve, \(\gamma : [a, b] \to \Omega, \gamma(t) = (x(t), y(t))\) and a 1-form \(\omega\) in \(\Omega\), one defines the line integral

\[
\int_{\gamma} \omega = \int_{a}^{b} P(x(t), y(t))x'(t)dt + P(x(t), y(t))y'(t)dt.
\]

The 1-form \(\omega\) is called \textit{closed} (in \(\Omega\)) if \(P_y(x, y) = Q_x(x, y)\) for all \((x, y) \in \Omega\). It is called \textit{exact} if there is a \(C^2\) real-valued function \(\psi(x, y)\) defined in \(\Omega\) such that \(\psi_x = P\), and \(\psi_y = Q\). That is, we can formally write

\[
d\psi = \psi_x dx + \psi_y dy = Pdx + Qdy
\]

It follows from the equality of mixed partial derivatives of \(\psi\) that every exact 1-form is closed. The converse is true if the region is simply connected.

It is proved in the calculus of several variables that, in a simply connected region, the line integral \(\int_{\gamma} \omega\) of a closed 1-form is \textit{path-independent}. Indeed, letting \(\psi\) be the function such that \(d\psi = \omega\), if \(\gamma_1\) and \(\gamma_2\) are two paths starting at the point \(p_1\) and ending at the point \(p_2\), then

\[
\int_{\gamma_1} \omega = \int_{\gamma_2} \omega = \psi(p_2) - \psi(p_1).
\]

Now, let \(\Omega\) be an open connected subset of the plane \(\mathbb{R}^2\), and let \(\eta = (\eta_1, \eta_2)\) be a \(C^2\) non-vanishing vector field defined in \(\Omega\).
Consider the 1-form

\[ \alpha_\eta = \frac{\eta_1 d\eta_2 - \eta_2 d\eta_1}{\eta_1^2 + \eta_2^2} = \frac{\eta_1(\eta_2 x dx + \eta_2 y dy) - \eta_2(\eta_1 x dx + \eta_1 y dy)}{\eta_1^2 + \eta_2^2} \]

In a region where \( \eta_1 \neq 0 \), this is

\[ d \left( \arctan \frac{\eta_2}{\eta_1} \right) \]

while in a region where \( \eta_2 \neq 0 \), this is

\[ -d \left( \arctan \frac{\eta_1}{\eta_2} \right) = d \left( \arccot \frac{\eta_2}{\eta_1} \right). \]

If \( \gamma : [0, 1] \to \Omega \) is a curve and \( \eta \) does not vanish on the image of \( \gamma \), then

\[ \int_\gamma \alpha_\eta \]

gives the \textit{total change in the angle that \( \eta \) makes with the positive horizontal direction} as one moves along the curve \( \gamma \). Since \( \eta \) points in the same direction at \( \gamma(0) \) and \( \gamma(1) \), this must be an integer multiple of \( 2\pi \).

We define the \textit{index} of \( \eta \) over \( \gamma \) to be the integer

\[ \text{Ind}(\gamma, \eta) = \frac{1}{2\pi} \int_\gamma \alpha_\eta \]
Definition. The standard parametrization of the unit circle $S^1$ is the map $\gamma: [0, 1] \to S^1$ defined by

$$\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$$

Exercises:

1. Using the standard parametrization $\gamma$ of the unit circle $S^1$, compute the index $\text{Ind}(\gamma, \eta)$ of each of the following vector fields.

(a) $\eta(x, y) = x \partial_x + y \partial_y$
(b) $\eta(x, y) = x \partial_x - y \partial_y$
(c) $\eta(x, y) = -x \partial_x - y \partial_y$

(d) Given a complex number $z = x + iy$, with $i = \sqrt{-1}$, we let $\text{Re}(z) = x, \text{Im}(z) = y$ denote its real and imaginary parts. Let $n$ be an integer, and consider the vector field

$$\eta(z) = z^n$$

(i.e., $\eta(x, y) = \text{Re}(z^n) \partial_x + \text{Im}(z^n) \partial_y$).

Compute $\text{Ind}(\gamma, \eta)$.

2. Let $\Omega$ be a region in the plane, and let $[a, b]$ and $[a_1, b_1]$ be two closed intervals in $\mathbb{R}$. Let $\gamma : [a, b] \to \Omega$ and $\gamma_1 : [a_1, b_1] \to \Omega$ be two $C^2$ curves. We say that $\gamma_1$ is a reparametrization of $\gamma$ if there is $C^2$ map $\rho : [a_1, b_1] \to [a, b]$ such that
(a) \( \rho(a_1) = a \) and \( \rho(b_1) = b \),
(b) \( \rho \) is strictly increasing (i.e., \( a_1 \leq s < t \leq b_1 \)
implies that \( \rho(s) < \rho(t) \)),
(c) \( \rho'(t) = \frac{D\rho}{dt}(t) \neq 0 \) for \( t \in [a_1, b_1] \), and
(d) \( \gamma_1 = \gamma \circ \rho \).

Let \( \omega = Pdx + Qdy \) be a \( C^2 \) 1-form in \( \Omega \). Show that
if \( \gamma_1 \) is a reparametrization of \( \gamma \) then
\[
\int_{\gamma_1} \omega = \int_{\gamma} \omega
\]

**Proposition 1.** Let \( \Omega \) be a region in the plane and
let \( \eta \) be a non-vanishing vector field in \( \Omega \). Let \( \gamma_0 \) and
\( \gamma_1 \) be two loops in \( \Omega \) which are \( C^2 \) homotopic through
loops in \( \Omega \). Then,
\[
\text{Ind}(\gamma_0, \eta) = \text{Ind}(\gamma_1, \eta)
\]

**Proof.** Let \( F(t, s) \) be a \( C^2 \) homotopy from \( \gamma_0 \) to \( \gamma_1 \)
so that \( F(t, 0) = \gamma_0 \) and \( F(t, 1) = \gamma_1(t) \) and \( F(0, s) = F(1, s) \) for all \( s \in [0, 1] \).

Let \( \gamma_s(t) = F(t, s) \) for each \( t, s \). Then, each \( \gamma_s \) is a
loop in \( \Omega \) so \( \text{Ind}(\gamma_s, \eta) \) is an integer for each \( s \). But,
\( \text{Ind}(\gamma_s, \eta) \) is an integral \( \int_0^1 \psi(s, t) dt \) in which the func-
tion \( \psi(s, t) \) is continuous. So, \( \text{Ind}(\gamma_s, \eta) \) depends con-
tinuously on \( s \). Since it is integer valued, it must be constant. QED.
Proposition 2. Let $\Omega$ be a region in the plane and let $\gamma$ be a loop in $\Gamma$. Let $\eta_1, \eta_2$ be two $C^2$ non-vanishing vector fields in $\Omega$ so that $\eta_1$ is $C^2$ homotopic to $\eta_2$ through non-vanishing vector fields in $\Omega$. Then,

$$\text{Ind}(\gamma, \eta_1) = \text{Ind}(\gamma, \eta_2)$$

Proof. This is similar to the preceding proof. Let $F(x, t)$ be the homotopy from $\eta_1$ to $\eta_2$ through non-vanishing vector fields. That is, $F(x, t)$ is a $C^2$ map from $\Omega \times [0, 1]$ to $\Omega$ such that $F(x, 0) = \eta_1(x)$ and $F(x, 1) = \eta_2(x)$ for each $x \in \Omega$. Letting $F_s(x) = F(x, s)$, the map $s \rightarrow F_s$ is a $C^2$ map from $[0, 1]$ to $R^2 \setminus \{0\}$.

Hence, the function $s \rightarrow \text{Ind}(\gamma, F_s)$ is continuous. Again, since it is integer valued, it must be constant. QED.

Theorem 3. Let $\Omega$ be a simply connected region in the plane. Let $\eta$ be a vector field in $\Omega$ and suppose that there as a loop $\gamma$ in $\Omega$ such that

$$\text{Ind}(\gamma, \eta) \neq 0.$$ 

Then, $\eta$ has a critical point in $\Omega$.

Proof.

Assume that $\eta$ has no critical point in $\Omega$, and let $p_0 \in \Omega$. Then, there is an $\epsilon > 0$ such that $B = B_\epsilon(p_0) \subset \Omega$. If
ε is small enough then η is nearly constant in B, so, any loop γ₁ whose image is in B has Ind(γ₁, η) = 0. But, simple connectivity gives that γ ≃ γ₁ which implies that Ind(γ, η) = 0. This contradiction proves the theorem. QED.

**Remark** One can generalize the notion of Ind(γ, η) where γ and η are only continuous. This can most easily be done through $C^2$ approximations. Suppose that γ : [0, 1] → Ω is a continuous loop in Ω and η : Ω → $\mathbb{R}^2$ is a continuous non-vanishing vector field in Ω. Then, given ε > there exist a $C^2$ loop γ₁ : [0, 1] → Ω such that |γ(t) − γ₁(t)| < ε for all t ∈ [0, 1]. For ε small enough, if γ₂ is another $C^2$ loop within ε of γ, then γ₁ ≃ γ₂ in Ω. The closure of the image G of a homotopy between γ₁ and γ₂ is a compact subset of Ω. There is a $C^2$ vector field η₁ in Ω such that |η(x) − η₁(x)| < ε for x ∈ G. We have defined Ind(γ₁, η₁). If we choose another $C^2$ loop γ₂ $C^0$ near γ and another $C^2$ vector field η₂ $C^0$ near η, then our invariance under homotopies shows that Ind(γ₁, η₁) = Ind(γ₂, η₁) = Ind(γ₂, η₂). Thus, the index is independent of the smooth approximations of γ and η. So, we can define Ind(γ, η) using any two sufficiently close $C^2$ approximations. The properties in the previous Propositions 1 and 2 and Theorem 3 remain valid.