# Differential Equations on the two dimensional torus

## 1 Constant vector fields on the torus

We will now study some simple differential equations on a two-dimensional torus. These will provide interesting minimal sets.

The 2−torus $\mathbf{T}^2$ is the product $S^1 \times S^1$ of two circles.

Writing $S^1$ as

$$S^1 = \{z \in \mathbf{C} : \mid z \mid = 1\},$$

we get

$$S^1 \times S^1 = \{(z_1, z_2) \in \mathbf{C}^2 : \mid z_1 \mid = \mid z_2 \mid = 1.$$

There is an alternative viewpoint which is useful.

Consider the quotient group $\mathbf{R}/\mathbf{Z}$; i.e., the set of equivalence classes in $\mathbf{R}$ under the equivalence relation $x \sim y$ iff $x - y \in Z$. There is a natural map $\pi : \mathbf{R} \to \mathbf{Z}$ assigning to each real number $x$ its equivalence class $[x]$.

Notice that the map $\phi : t \to e^{2\pi i t}$ is a surjective map from $\mathbf{R}$ onto $S^1$ such that each pre-image $\phi^{-1}(z)$ is an equivalence class in $\mathbf{R}/\mathbf{Z}$. Thus, the map $\psi([x]) = \phi(x)$ gives a well-defined bijection from $\mathbf{R}/\mathbf{Z}$ to $S^1$.

We can use the map $\psi$ to define a topology on $\mathbf{R}/\mathbf{Z}$, so we have notions of continuity, etc.

Note that we can also think of the circle as a closed interval $[a, b]$ with its endpoints identified.

We can now apply these ideas to the product $T^2 = S^1 \times S^1$. There is a bijection from $\mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$ onto $T^2$. This defines a topology on $T^2$. There is another natural bijection between $\mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$ and $\mathbf{R}^2/\mathbf{Z}^2$, and we can think of this latter quotient set as $\mathbf{T}^2$ as well. We will use all of these objects.

In particular, we think of $T^2$ as either $S^1 \times S^1$ or $\mathbf{R}^2/\mathbf{Z}^2$.

Now we wish to define differential equations on $\mathbf{T}^2$. We will use the representation $\mathbf{R}^2/\mathbf{Z}^2$.

For this purpose, let $f(x, y), g(x, y)$ be two real-valued functions of two real variables such that they are periodic of period 1 in both variables. That is,

$$f(x+1, y+1) = f(x,y), \ g(x+1, y+1) = g(x,y), \quad \forall x, y.$$

The planar vector field $X(x,y) = (f(x,y), g(x,y))$ is a map from $\mathbf{R}^2 \to \mathbf{R}^2$. The periodicity conditions imply that it induces a well-defined map from $\mathbf{R}^2/Z^2$ to $R^2$. We call this a vector field on the torus $\mathbf{T}^2$. The associated system of differential equations

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

is called a *differential equation on* $\mathbf{T}^2$.

The solutions (or orbits) are obtained by taking the orbits in $\mathbf{R}^2$ and projecting them down to $\mathbf{T}^2$.

Let us take a simple example.

Consider the constant vector field in $\mathbf{R}^2$

$$X(x,y) = \omega_1 \partial_x + \omega_2 \partial_y,$$

where $\omega_1, \omega_2$ are positive real numbers.

The associated system of differential equations is

$$\begin{aligned} \dot{x} &= \omega_1 \\ \dot{y} &= \omega_2 \end{aligned} \tag{1}$$

The periodicity conditions are obviously satisfied, so we get a differential equation on $\mathbf{T}^2$. This is called the *constant* or *linear* vector field on $\mathbf{T}^2$.

Let's us consider its orbits.

The solutions to (1) in $\mathbf{R}^2$ are the lines

$$x(t) = \omega_1 t + c_1, \ y(t) = \omega_2 t + c_2.$$

Each such line has slope $\frac{\omega_2}{\omega_1}$ and passes through the point $(c_1, c_2)$.

On the torus $\mathbf{T}^2$ we simply take $(x(t), y(t))$ mod 1.

We have two main cases:

1. $\frac{\omega_2}{\omega_1}$ is rational. Write it as $\frac{\omega_2}{\omega_1} = \frac{p}{q}$ in lowest terms with $p, q$ positive integers.

   Consider the orbit $\gamma_0$ in $\mathbf{R}^2$ through $(0,0)$ (i.e., $c_1 = c_2 = 0$).

The solution in $\mathbf{R}^2$ has the form

$$x(t) = \omega_1 t, \quad y(t) = \left(\frac{p}{q}\right) \omega_1 t$$

Let $\pi : \mathbf{R}^2 \to \mathbf{T}^2$ denote the canonical projection.

When $t = q/\omega_1$, we have $x(t) = q,\ y(t) = p$, so the projection $\pi(\gamma_0) = \gamma$ into $\mathbf{T}^2$ is a closed curve (topological circle) in $\mathbf{T}^2$.

Since all the solutions are vertical tranlates of $\gamma_0$ we have that all solutions in $\mathbf{T}^2$ are periodic of the same period.

2. $\frac{\omega_2}{\omega_1}$ is irrational.

   This case is more interesting. We will show that every orbit in $\mathbf{T}^2$ is dense in $\mathbf{T}^2$. Hence, all of $\mathbf{T}^2$ is a minimal set.

   Let us show that the orbit through $(0, 0)$ is dense in $\mathbf{T}^2$. The analogous statement for other orbits is similar.

   Consider the circle $S^1 = \pi(\{x = 0\})$.

   There is a first return map $F : S^1 \to S^1$ (also called Poincaré map) obtained by taking a point $(0, y)$ and taking the point $(1, F(y))$ where the orbit through $(0, y)$ hits the line $\{x = 1\}$.

   Let us compute $F$.

   The orbit through $(0, y)$ is $\{(\omega_1 t, \omega_2 t + y),\ t \in \mathbf{R}\}$.

   We get

$$\begin{array}{rcl} \omega_1 t & = & 1 \\ \omega_2 t + y & = & F(y). \end{array}$$

   Hence,

$$t = 1/\omega_1, \quad F(y) = y + \omega_2/\omega_1$$

   So, letting $\alpha = \frac{\omega_2}{\omega_1}$ we get that $F(y) = y + \alpha$ $(mod\ 1)$.

   To show that the orbits of $X$ are dense, it suffices to show that the iterates $\{F^n(y), n \in \mathbf{Z}\}$ are dense mod 1

This is the number theoretic statement: for $\alpha$ irrational, and

$$A = \{n\alpha + m : n \in \mathbf{Z}, m \in \mathbf{Z}\},$$

then

$$A \text{ is dense in } \mathbf{R}. \tag{2}$$

This elementary fact is proved as follows.

(a) $A$ is clearly an additive subgroup of $\mathbf{R}$.

(b) To show that $A$ is dense, it suffices to show that 0 is an accumulation point of $A$.

(c) For this latter fact, for each $n \geq 0$, let $m_n \in \mathbf{Z}$ be such that $x_n = n\alpha + m_n \in [0, 1)$. Since $\alpha$ is irrational, the numbers $x_n$ are all distinct. Since $[0, 1]$ is compact, there is a subsequence $x_{n_i}$ of $(x_n)$ which converges. In particular, if $\epsilon > 0$ is arbitrary, there are points $x_{n_1}, x_{n_2}$ such that

$$| \, x_{n_1} - x_{n_2} \, | < \epsilon.$$

But then this difference $x_{n_1} - x_{n_2}$ is in $A \bigcap (-\epsilon, \epsilon)$. Since, $\epsilon$ is arbitrary, we have that 0 is an accumulation point of $A$ as required.

Let us now give an application of these ideas.

Consider the mass spring system whose equation is

$$m\ddot{x} + kx = 0$$

where $m$ is the mass of an object suspended vertically by a spring, $k > 0$ is the spring constant, and $x = 0$ represents the equilibrium position.

Let $\omega = \sqrt{\frac{k}{m}}$. The associated first order planar system can be written

$$\begin{aligned} \dot{x} &= -\omega \, y \\ \dot{y} &= \omega \, x \end{aligned} \tag{3}$$

Let us write this system (3) in polar coordinates.

We have

$$x = rcos(\theta),\ \ y = rsin(\theta)$$

$$\begin{aligned} \dot{x} &= \dot{r}cos(\theta) - rsin(\theta)\dot{\theta} \\ \dot{y} &= \dot{r}sin(\theta) + rcos(\theta)\dot{\theta} \end{aligned} \tag{4}$$

So, we see that the simple polar coordinate system

$$\begin{aligned} \dot{r} &= 0 \\ \dot{\theta} &= \omega \end{aligned} \tag{5}$$

is equivalent to (4).
The solutions off $(0,0)$ are

$$r = c > 0, \quad \theta(t) = \omega\ t + \theta_0.$$

Hence, the orbits in the plane off $(0,0)$ consist of the circles $r = c$ and the motion has constant rotational speed $\omega$ with period $\frac{2\pi}{\omega}$.

Now, consider two uncoupled springs whose equations in $\mathbf{R}^4$ become

$$\begin{aligned} \dot{x_1} &= -\omega_1\ y_1 \\ \dot{y_1} &= \ \omega\ x_1 \\ \dot{x_2} &= -\omega_2\ y_2 \\ \dot{y_2} &= \ \omega_2\ x_2 \end{aligned} \tag{6}$$

Using polar coordinates $(r_1, \theta_1, r_2, \theta_2)$ in $\mathbf{R}^4$ in the obvious way, we get the equivalent equations

$$\begin{aligned} \dot{r_1} &= 0 \\ \dot{\theta_1} &= \omega_1 \\ \dot{r_2} &= 0 \\ \dot{\theta_2} &= \omega_2 \end{aligned} \tag{7}$$

Hence all solutions in $\mathbf{R}^4$ off the union of the two subspaces $\mathbf{R}^2 \times \{(0,0\} \bigcup \{0,0\} \times \mathbf{R}^2$ lie on two dimensional tori $r_1 = c_1, r_2 = c_2$, and the motion on each is a constant vector field. If $\omega_2/\omega_1$ is rational, then all these orbits are periodic, and if $\omega_2/\omega_1$ is irrational, then all these tori are minimal sets.

# 2 Gradient vector fields on the torus

To specify a gradient vector field on the two dimensional torus, we need to consider smooth potential functions on the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. These are smooth functions $\psi(x, y)$ such that

$$\psi(x + 1, y + 1) = \psi(x, y) \text{ for all } (x, y) \in \mathbf{R}^2.$$

From the theory of Fourier series, we can write such functions as sums

$$\psi(x, y) = \sum_{k=0}^{\infty} a_k cos(2\pi kx) + b_k sin(2\pi k)$$

The simplest such examples are the trigonometric polynomials

$$\psi_N(x, y) = \sum_{k=0}^{N} a_k cos(2\pi kx) + b_k sin(2\pi k)$$

for a positive integer $N$.

Givena function $\psi(x, y)$ with $\psi(x + 1, y + 1) = \psi(x, y)$, we obtain its gradient vector field by

$$X(x, y) = (\partial_x \psi(x, y), \partial_y \psi(x, y))$$

We wish to give a simple geometric interpretation of this in a special case.

First, we recall some concepts about surfaces of revolution in $\mathbf{R}^3$.

Let $(X, Y, Z)$ denote the coordinates of a point in $\mathbf{R}^3$.

Let $I = (a, b)$ be an open interval in $\mathbf{R}$ with $a < b$.

Consider a $C^r$ parametrized curve $\eta : I \to \mathbf{R}^3$ in the $XY$-plane given by

$$\eta(t) = (X(t), Y(t), 0)$$

such that

$$\inf_{t \in I} \mid X(t) \mid) > 0$$

and

$$\inf_{t \in I} \mid X'(t)^2 + Y'(t)^2 \mid > 0$$

with $r \geq 1$.

Thus, the image of $\eta$ is disjoint from the $Y-$ axis, and the tangent vectors to $\eta$ never vanish.

The surface obtained by revolving $\eta$ about the $Y-$axis can be parametrized by

$$\gamma(t, \theta) = (X(t)cos(2\pi\theta), Y(t), X(t)sin(2\pi\theta))$$

with $0 < \theta < 1$).

Thus, the map $\gamma : (a, b)\times(0, 1) \to \mathbf{R}^3$ is a $C^r$ embedding from $(a, b)\times(0, 1)$ into $\mathbf{R}^3$. Its image is the surface of revolution determined by $\eta$ and the $Y-$ axis.

Next, consider the open unit box in $\mathbf{R}^2$ given by $(0, 1) \times (0, 1)$. We use this to parametrize an open dense set of a torus of revolution in $\mathbf{R}^3$ as follows.

Let $0 < r < R$ be two positive real numbers.

Consider the embedding $\eta$ from $(0, 1)$ into the $X - Y-$plane given by

$$\eta(t) = (R + r\ cos(2\pi t), r\ sin(2\pi t), 0) = (X(t), Y(t), 0)$$

This maps $(0, 1)$ onto the circle

$$\{(X, Y, 0) : (X - R)^2 + Y^2 = r^2\}$$

with a point removed.

Using this to parametrize the surface obtained by revolving $\eta$ about the $Y-$axis, we get the map $\Phi : (0, 1) \times (0, 1) \to \mathbf{R}^3$ defined by

$$\Phi(t, \theta) = ((R + rcos(2\pi t)cos(2\pi\theta), r\ sin(2\pi t), (R + rcos(2\pi t)sin(2\pi\theta)).$$

The image is an open torus of revolution with two circles removed. The image is dense in the torus of revolution.

Consider the *height function* $h(X, Y, Z) = Z$ from $\mathbf{R}^3$ to $\mathbf{R}$. This function is defined as a $C^\infty$ function on all of $\mathbf{R}^3$, and, in particular on the closure of the image of $\Phi$.

This gives a function $\psi(t, \theta)$ from $(0, 1) \times (0, 1)$ to $\mathbf{R}$ defined by

$$\psi(t, \theta) = (R + rcos(2\pi t))sin(2\pi\theta)$$

Replacing $t$ by $x$ and $\theta$ by $y$, and still using $\psi$ to denote the function, we have

$$\psi(x,y) = (R + rcos(2\pi x))sin(2\pi y). \tag{8}$$

This formula can be extended to a doubly periodic (periodic in both $x$ and $y$) function from $\mathbf{R}^2$ to $\mathbf{R}$ which induces a function $\tilde{\psi} : \mathbf{T}^2 \to \mathbf{R}$.

Let $\Sigma$ denote the closure of the image of $\Phi$. This is a representation of a torus of revolution in $\mathbf{R}^3$. The function $h$ restricted to $\Sigma$ is (by definition) a $C^\infty$ function on $\Sigma$. Consider the level sets of $h$ in $\Sigma$; i.e., the intersections of $\{h(X,Y,Z) = c\}$ and $\Sigma$. These sets are one of the following types.

1. isolated points
2. disjoint union of two topological circles
3. a topological figure eight
4. topological circles

For $p \in \Sigma$, let $T_p\Sigma$ denote the tangent plane to $\Sigma$ at $p$.

One can define a vector field on $\Sigma$ by assigning to each point $p \in \Sigma$ the vector in $T_p\Sigma$ obtained by projecting the vector $(0,0,1)$ at $p$ into $T_p\Sigma$. This is defined to by the gradient vector field on $\Sigma$ induced by the function $h \mid \Sigma$.

Using the parametrization $\Phi : (0,1) \times (0,1) \to \Sigma$ this vector field can be pulled back to $(0,1) \times (0,1)$. This latter vector field can be extended to all of $\mathbf{R}^2$, and this is the gradient vector field of the function $\psi$ above.

**Remark** The function $\psi$ is a real-valued function from $\mathbf{R}^2$ to $\mathbf{R}$ with infinitely many critical points. Its level sets in $(0,1) \times (0,1)$ are the pre-images by $\Phi$ of the level sets of $h$ on the image of $\Phi$.