We have proved that solutions to differential equations depend continuously on parameters.

Now we wish to investigate the smooth dependence of solutions in systems which depend smoothly on parameters.

Theorem. *Suppose that $f(t,x,\lambda)$ is a C^1 function of the variables (t,x,λ) in an open set $D \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$. For, $(t_0,x_0,\lambda_0) \in D$, let $\phi(t,t_0,x_0,\lambda)$ be the solution of the initial value problem*

$$\dot{x} = f(t,x,\lambda), \quad x(t_0) = x_0 \tag{1}$$

.

Then, the solution $\phi(t,u,x,\lambda)$ to the initial value problem

$$\dot{x} = f(t,x,\lambda), \quad x(u) = x \tag{2}$$

is a C^1 function of the variables (t,u,x,λ) for (u,x,λ) near (t_0,x_0,λ_0). Moreover, the $x-$partial derivative of the solution

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x}(t,u,x,\lambda) \equiv J(t)$$

with respect to the space variable x satisfies the matrix (operator) initial value problem

$$\dot{J} = \frac{\partial f}{\partial x}(t,\phi(t,u,x,\lambda),\lambda) \cdot J, \quad J(u) = id \tag{3}$$

Remark. The differential equation in (3) (without the λ parameter) is usually called the *variational equation* of (1).

We will leave the proof of this last theorem and some of its generalizations to the exercises.

Differential Inequalities

Let D_r denote the right hand derivative of a function; i.e.,

$$D_r f(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$

If $\omega(t,u)$ is a scalar function for the scalar variables t,u in some open connected set Ω, we say a function $v(t)$ is a *solution of the differential inequality*

$$D_r v(t) \leq \omega(t, v(t)) \tag{4}$$

on $[a,b)$ if $v(t)$ is continuous on $[a,b)$, right differentiable on $[a,b)$ and satisfies (4) on $[a,b)$.

Lemma. *If $x(\cdot)$ is a continuously differentiable map from a closed interval $[a,b]$ in $\mathbf{R}$, then $D_r| x(t) |$ exists on $[a,b)$ and $| D_r(| x(t) |) | \leq | \dot{x}(t) |$ for all $t \in [a,b)$.*

Proof.

For any two $n-$ vectors, x, u, and $0 < \tau \leq 1, 0 < h$, we have

$$| x + \tau h u | - | \tau x + \tau h u | \leq (1-\tau)| x |$$

or

$$| x + \tau h u | - | x | \leq \tau(| x + hu | - | x |)$$

or

$$\frac{| x + \tau h u | - | x |}{\tau h} \leq \frac{| x + hu | - | x |}{h}$$

This implies that the difference quotient

$$\phi(h) = \frac{| x + hu | - | x |}{h}$$

is a non-decreasing function of h.

Also,

$$\begin{aligned} \phi(h) &= \frac{| x + hu | - | x |}{h} \\ &= \frac{| x - (-hu) | - | x |}{h} \\ &\geq \frac{| x | - | -hu | - | x |}{h} \\ &\geq -| u |, \end{aligned}$$

so, $\phi(h)$ is bounded below.

Thus,

$$\lim_{h \to 0+} \phi(h)$$

exists.
Now, suppose $x(t)$ is C^1.
then,

$$\begin{aligned} \Big| \ (| \, x(t+h) \, | - | \, x(t) \, |) \ &- \ (| \, x(t) + h\dot{x}(t) \, | - | \, x(t) \, |) \ \Big| \\ &= \Big| \ | \, x(t+h) \, | - | \, x(t) + h\dot{x}(t) \, | \ \Big| \\ &\leq \ | \, x(t+h) - x(t) - h\dot{x}(t) \, | = o(h) \end{aligned}$$

as $h \to 0+$.
Thus, $D_r(| \, x(t) \, |)$ exists and equals

$$\lim_{h \to 0+} \frac{| \, x(t) + h\dot{x}(t) \, | - | \, x(t) \, |}{h}$$

Taking the norm of both sides gives $| \, D_r(| \, x(t) \, |) \, | \leq | \, \dot{x}(t) \, |$ which proves the Lemma. QED.

For the sequel, it will be convenient to make some definitions.

A *regular* $n-$vector field is a pair (f, D) in which D is an open subset of $\mathbf{R} \times \mathbf{R}^n$ and f is a map from D into $\mathbf{R}^n$ which satisfies the following conditions.

1. f is continuous.

2. For each $(t_0, x_0) \in D$ there are an open set $U \subset D$ and a constant $K = K(U) > 0$ such that $(t_0, x_0) \in U$ and if (t, x) and (t, y) are two points in U with the same first coordinate, then

$$| \, f(t, x) - f(t, y) \, | \leq K | \, x - y \, |$$

The set D is called the *domain* of f.

We also say that f is an $n-$regular vector field with domain D.

As in the case of metric spaces, we will often drop the domain D and simply say that f is an $n-$regular vector field.

Thus, if we use coordinates (t, x) on D, an $n-$regular vector field f si simply a continuous $n-$vector valued map f which is locally Lipschitz in x on D as we have defined before.

We call a $1-$regular vector field f a *regular scalar field.* Thus, the domain of such an f is a subset of $\mathbf{R} \times \mathbf{R}$ and the range is a subset of $\mathbf{R}$.

Since we consider regular $n-$vector fields (f, D) and certain associated regular scalar fields, we use the notation (ω, Λ) for the scalar fields. In this case, the map ω will take the open subset $\Lambda \subseteq \mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$.

With our new definitions, it is convenient to introduce new terminology for solutions to initial value problems for $n-$regular vector fields.

Given an $n-$regular vector field (f, D), and a point $(t_0, x_0) \in D$, a (t_0, x_0, f) *integral curve* is a C^1 map $\gamma : I \to \mathbf{R}^n$ from an open real interval I about t_0 in $\mathbf{R}$ into $\mathbf{R}^n$ such that

1. $\gamma(t_0) = x_0$,
2. $(t, \gamma(t)) \in D$ for every $t \in I$, and
3. $\dot{\gamma}(t) = f(t, \gamma(t))$ for every $t \in I$.

Thus, a $(t_0, x_0, f)-$integral curve is simply a solution to the initial value problem

$$\dot{x} = f(t, x), \;\; x(t_0) = x_0$$

We use the terminology γ *is an integral curve for* f to mean that γ is a $(t_0, x_0, f)-$integral curve for some (t_0, x_0).

A *maximal* $(t_0, x_0, f)-$integral curve γ_0 is one with domain (a, b) with $-\infty \le a < b \le \infty$ so that if γ is any $(t_0, x_0, f)-$ integral curve with domain I, then $I \subseteq (a, b)$ and $\gamma(t) = \gamma_0(t)$ for $t \in I$.

When γ is a $(t_0, x_0, f)-$integral curve we often say that γ is an integral curve for f *through* (t_0, x_0) or *about* (t_0, x_0).

In addition, we can make the analogous obvious definitions of *right* integral curves, *right maximal integral curves*, *left integral curves*, and *left maximal integral curves*, where we consider intervals of the form $[t_0, b)$ or $(a, t_0]$ respectively, and we use one-sided derivatives at the point t_0.

Theorem. *Let* (ω, Λ) *be a regular scalar field with domain* $\Lambda \subset \mathbf{R} \times \mathbf{R}$. *Suppose* $u(\cdot)$ *is an integral curve of* ω *defined on the the interval* $[a, b]$ *and* $v(t)$ *is a solution of the differential inequality (4) on* $[a, b)$ *with* $v(a) \le u(a)$. *Then,* $v(t) \le u(t)$ *for* $t \in [a, b)$.

Proof.

Consider the sequence of equations

$$\dot{u} = \omega(t, u) + \frac{1}{n} \tag{5}$$

for $n = 1, 2, \ldots$.

Let $u_n(t)$ denote the solution to the equation $\dot{u} = \omega(t, u) + \frac{1}{n}$ with $u_n(a) = u(a)$. By the global continuity theorem, there is an $n_0 > 0$ such that for $n \geq n_0$, the solution $u_n(t)$ is defined on all of $[a, b]$ and converges uniformly to $u(t)$ as $n \to \infty$.

We show that

(*) $v(t) \leq u_n(t)$ for all $t \in [a, b]$ and $n \geq n_0$

Once this is done, since u_n converges uniformly to u on $[a, b]$, it follows that $v(t) \leq u(t)$ for all t.

If (*) fails for some $n \geq n_0$, then then there exist $t_2 < t_1$ in (a, b) such that $v(t) > u_n(t)$ on $(t_2, t_1]$ and $v(t_2) = u_n(t_2)$. Therefore, $v(t) - v(t_2) > u_n(t) - u_n(t_2)$ on $(t_2, t_1]$.

This implies that

$$\begin{aligned} D_r v(t_2) &\geq \dot{u}_n(t_2) = \omega(t_2, u_n(t_2)) + \frac{1}{n} \\ &= \omega(t_2, v(t_2)) + \frac{1}{n} \\ &> \omega(t_2, v(t_2)) \end{aligned}$$

which contradicts the assumption that v is a solution to (4). QED

Corollary. *Suppose (ω, Λ) is a regular scalar field with integral curve $u(\cdot)$ whose domain contains the closed interval $[a, b]$. If $x(t)$ is a C^1 $n-$vector valued function on $[a, b]$ such that $\mid x(a) \mid \leq u(a)$, and, for $t \in [a, b]$ we have $(t, \mid x(t) \mid) \in \Lambda$ and*

$$\mid \dot{x}(t) \mid \leq \omega(t, \mid x(t) \mid),$$

then $\mid x(t) \mid \leq u(t)$ for all $t \in [a, b]$.

Proof.

By a previous Lemma, we have $D_r(\mid x(t) \mid)$ exists and is no larger than $\mid \dot{x}(t) \mid$. Thus, $\mid x(t) \mid$ satisfies the differential inequality $D_r(v(t)) \leq \omega(t, v(t))$ on $[a, b]$. By the previous theorem, we have $\mid x(t) \mid \leq u(t)$ for all t. QED.

Definition. Let (f, D) be a regular $n-$vector field and let (ω, Λ) be a regular scalar vector field. We say that (λ, Λ) *norm-dominates* (or simply *dominates*) (f, D) if for any $(t, x) \in D$, we have

1. $(t, \mid x \mid) \in \Lambda$, and

2. $| f(t,x) | \leq \omega(t, | x |)$.

Proposition. *Suppose* (ω, Λ) *is a regular scalar field which dominates the regular* $n-$*vector field* (f, D)*, and* $u(\cdot)$ *is a non-negative integral curve of* ω *whose domain contains the interval* $[a, b)$*. Assume that the set*

$$E = \{(t, y) : t \in [a, b), 0 \leq y \leq u(t)\}$$

is contained in Λ*.*

Let γ *be a right maximal integral curve of* f *with domain* $[a, b_1)$ *such that* $| \gamma(a) | \leq u(a)$*.*

Then, either $(t, \gamma(t))$ *approaches the boundary of* D *as* t *approaches* b_1 *from the left or* $[a, b) \subseteq [a, b_1)$*. In the latter case, we also have* $| \gamma(t) | \leq u(t)$ *for all* $t \in [a, b)$*.*

Proof.

If $(t, \gamma(t))$ does not approach the boundary of D, then the only way the $(a, \gamma(a), f)-$integral curve can fall to exist on $[a, b)$ is that there is some $c \in (a, b)$ such that $\gamma(t)$ is defined for $a \leq t < c$ and

$$\limsup_{t \to c-} | f(t, \gamma(t)) | = \infty. \tag{6}$$

Now, $u(t)$ exists for all $t \in [a, c]$ and

$$E \stackrel{\text{def}}{=} \{(t, y) : t \in [a, c], 0 \leq y \leq u(t)\}$$

is a compact subset of Λ. So, $\omega(t, y)$ is bounded on E. Thus, there is a constant $K > 0$ so that $\omega(t, y) \leq K$ for all $(t, y) \in E$.

Let $c_1 \in (a, c)$ be such that

$$| f(c_1, \gamma(c_1)) | > K. \tag{7}$$

Now, the domination condition gives

$$D_r(| \gamma(t) |) \leq | \dot{\gamma}(t) | = | f(t, \gamma(t)) | \leq \omega(t, | \gamma(t) |), | \gamma(a) | \leq u(a),$$

and both $u(t)$ and $\gamma(t)$ exist for all $t \in [a, c_1]$.

So, by the Corollary above, we have that $| \gamma(t) | \leq u(t)$ for all $t \in [a, c_1]$.

Thus,

$$F \stackrel{\text{def}}{=} \{(t, | \gamma(t) |) : t \in [a, c_1]\} \subseteq E$$

and, in particular, $\mid f(c_1, \gamma(c_1)) \mid \leq \omega(c_1, \mid \gamma(c_1) \mid) \leq K$ which contradicts (7).

This proves the theorem. QED.

Now we give a result on the existence of long time solutions to some special regular scalar fields.

Proposition. *Consider the regular scalar field* (ω, Λ) *where* $\omega(t, u) = \phi(t)\psi(u)$ *on the region* $(\alpha, \infty) \times (a, \infty)$ *(this implies that* ϕ *is continuous, and* ψ *is locally Lipschitz) with* ϕ *non-negative on* (α, ∞) *and* ψ *positive on* (a, ∞). *If*

$$\int_a^\infty \frac{du}{\psi(u)} = \infty,$$

then, for any $t_0 > \alpha, u_0 > a$, *the right maximal* (t_0, u_0, ω) *integral curve is defined on* $[t_0, \infty)$.

Proof. Suppose the right maximal integral curve $u(\cdot)$ only exists on the interval $[t_0, c)$. If $u(t)$ remained bounded as $t \to c-$, then, continuity of ψ on closed bounded intervals would imply that $\phi(t)\psi(u(t))$ remains bounded on $[t_0, c)$. Consequently, $u(t)$ would have a limit as $t \to c-$ and then, we could get a non-tivial continuation of $u(t)$ to an interval $[t_0, c + \epsilon)$ for some positive ϵ. This would contradict the maximality, so there must exist a sequence $t_n \in [t_0, c)$ with $t_n \to c-$ and $u(t_n) \to \infty$.

Now, since $\frac{du}{dt}(t) = \phi(t)\psi(u(t))$, we have

$$\int_{u_0}^{u(t)} \frac{dv}{\psi(v)} = \int_{t_0}^{t} \phi(s)ds.$$

Putting t_n into this formula gives

$$\int_{u_0}^{u(t_n)} \frac{dv}{\psi(v)} = \int_{t_0}^{t_n} \phi(s)ds.$$

By assumption on ψ, the left hand side becomes infinite as $n \to \infty$, but the right hand side remains bounded by $\int_{t_0}^{c} \mid \phi(s) \mid ds$. This is a contradiction, so the solution $u(t)$ exists on all of $[t_0, \infty)$. QED

Theorem. *Let* $\omega(t, u) = \phi(t)\psi(u)$ *and* (t_0, u_0) *be as in the previous Proposition. Let* $(f, \mathbf{R} \times \mathbf{R}^n)$ *be an* $n-$*regular vector field defined on all of* $\mathbf{R} \times \mathbf{R}^n$ *which is dominated by* $(\omega, (\alpha, \infty) \times (a, \infty))$. *Then, any right maximal* $(t_0, x_0, f)-$*integral curve with* $\mid x_0 \mid \leq u_0$ *has the whole interval* $[t_0, \infty)$ *in its domain.*

Proof. This follows immediately from the preceding results. QED.

Corollary. *Suppose that $A(t)$ is an $n \times n$ matrix-valued function of t and $h(t)$ is an $n-$vector valued function of t and that both $A(t), h(t)$ are defined and continuous on the whole line $\mathbf{R}$. Let $x_0 \in \mathbf{R}^n$ and $t_0 \in \mathbf{R}$. Then, the linear differential equation*

$$\dot{x} = A(t)x + h(t)$$

has a unique solution $x(t)$ with $x(t_0) = x_0$ which exists for all t.

Proof. It suffices to bound

$| A(t)x + h(t) |$ by a function of the form $\phi(t)\psi(| u |)$ with ϕ continuous on $\mathbf{R}$, $\psi(u), C^1$ on $\mathbf{R}$, and $\int^{\infty} \frac{dv}{\psi(v)} = \infty$.

We can use $\phi(t) = \max(| A(t) |, | h(t) |)$ and $\psi(u) = u + 1$. QED