Dependence of Solutions to Differential Equations on Parameters

We now want to investigate the dependence of solutions to differential equations on parameters.

Lemma.(Gronwall Inequality). *Suppose* $f(t), a \leq t \leq b$ *is a continuous non-negative real-valued function on the closed real interval* $[a, b]$ *such that there are positive constants* K_1, K_2 *such that, for all* $t \in [a, b]$,

$$f(t) \leq K_1 + K_2 \int_a^t f(s)ds.$$

Then, for all $t \in [a, b]$,

$$f(t) \leq K_1 exp(K_2(t-a)) \leq K_1 exp(K_2(b-a))$$

Proof.

Let $U(t) = K_1 + K_2 \int_a^t f(s)ds$. Then, U is a strictly positive continuously differentiable function on $[a, b]$ with

$$U'(t) = K_2 f(t) \leq K_2 U(t)$$

for all t. Thus, $\frac{U'(t)}{U(t)} \leq K_2$. Integrating this last inequality over the interval $[a, t]$ gives

$$logU(t) - logU(a) \leq K_2(t-a)$$

or

$$logU(t) \leq logU(a) + K_2(t-a)$$

and

$$f(t) \leq U(t) \leq U(a)exp(K_2(t-a)) = K_1 exp(K_2(t-a))$$

QED.

Theorem 1 (Local continuity of solutions on parameters). *Suppose* $f(t, x, \lambda)$ *is a continuous function defined in an open set* $D \subseteq \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$. *Suppose that there are constants* $M > 0$, $K > 0$ *such that*

1. *for* $(t, x, \lambda) \in D$, $| f(t, x, \lambda) | \leq M$
2. *for* $(t, x, \lambda), (t, y, \lambda) \in D$, $| f(t, x, \lambda) - f(t, y, \lambda) | \leq K | x - y |$.

Let $(t_0, x_0, \lambda_0) \in D$. *Then, there are a positive number* $\alpha > 0$ *and a neighborhood* V *of* (t_0, x_0, λ_0) *such that for each* $(u, y, \lambda) \in V$, *the IVP* $\dot{x} = f(t, x, \lambda), x(u) = y$ *has a unique solution* $\phi(t, u, y, \lambda)$ *defined on the interval* $[u - \alpha, u + \alpha]$ *and the function* $\phi(t, u, y, \lambda)$ *is a continuous function of the variables* (t, u, y, λ) *in* $[t_0 - \alpha, t_0 + \alpha] \times V$.

Remark. This result says that for all (u, y, λ) near (t_0, x_0, λ_0) the solution to the IVP $\dot{x} = f(t, x, \lambda), x(u) = y$ is defined on the same sized interval (of length 2α) about the initial time u and the solution depends continuously on the initial time, value, and parameter.

Proof.

Let $I_{\alpha_0} = [t_0 - \alpha_0, t_0 + \alpha_0]$.

First take a closed product neighborhood $\bar{V} = I_{\alpha_0} \times B_\beta \times C_\gamma \subset \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$ where $\alpha_0 > 0, \beta > 0, \gamma > 0$ so that

$$\bar{V} \subseteq D$$

Then, in $\bar{V}$, we have $| f(t, x, \lambda) | \leq M$ and $| f(t, x, \lambda) - f(t, y, \lambda) | \leq K | x - y |$.

Let $\alpha \in (0, \alpha_0/2)$ be such that

$$\alpha M \leq \frac{\beta}{4} \tag{1}$$

and

$$\alpha K < 1. \tag{2}$$

Let $V = [t_0 - \alpha, t_0 + \alpha] \times B_{\frac{\beta}{2}} \times C_\gamma$.

Claim 1:

For $(u, y, \lambda) \in V$, the IVP, $\dot{x} = f(t, x, \lambda), x(u) = y$ has a unique solution $\phi(t, u, y, \lambda)$ defined on the interval $[u - \alpha, u + \alpha]$ and the vector $\phi(t, u, y, \lambda) \in B_\beta$.

This is proved exactly as the proof of the E-U Theorem and will be left as an exercise.

Claim 2: The solution $\phi(t,u,y,\lambda)$ is a continuous function on $[t_0-\alpha, t_0+\alpha]\times V$.

To prove this, we write, for $t \geq u$,

$$\mid \phi(t,u,y,\lambda) - \phi(\bar{t},\bar{u},\bar{y},\bar{\lambda}) \mid$$

$$\begin{aligned} &\leq \mid \phi(t,u,y,\lambda) - \phi(t,\bar{u},\bar{y},\bar{\lambda}) \mid + \mid \phi(t,\bar{u},\bar{y},\bar{\lambda}) - \phi(\bar{t},\bar{u},\bar{y},\bar{\lambda}) \mid \\ &\leq \mid \phi(t,u,y,\lambda) - \phi(t,\bar{u},\bar{y},\bar{\lambda}) \mid + M \mid t-\bar{t} \mid . \end{aligned}$$

Also, we have

$$\mid \phi(t,u,y,\lambda) - \phi(t,\bar{u},\bar{y},\bar{\lambda}) \mid$$

$$\begin{aligned} &\leq \mid y + \int_u^t f(s,\phi(s,u,y,\lambda),\lambda)ds \\ &\qquad -\bar{y} - \int_{\bar{u}}^t f(s,\phi(s,\bar{u},\bar{y},\bar{\lambda}),\bar{\lambda})ds \mid \\ &\leq \mid y-\bar{y} \mid + \int_u^t \mid f(s,\phi(s,u,y,\lambda),\lambda) - f(s,\phi(s,\bar{u},\bar{y},\bar{\lambda}),\bar{\lambda}) \mid ds \\ &\qquad + \mid \textstyle\int_u^t f(s,\phi(s,\bar{u},\bar{y},\bar{\lambda}),\bar{\lambda})ds - \int_{\bar{u}}^t f(s,\phi(s,\bar{u},\bar{y},\bar{\lambda}),\bar{\lambda})ds \mid \\ &\leq \mid y-\bar{y} \mid + M \mid u-\bar{u} \mid \\ &\qquad + \int_u^t \mid f(s,\phi(s,u,y,\lambda),\lambda) - f(s,\phi(s,u,y,\lambda),\bar{\lambda}) \mid ds \\ &\qquad + \int_u^t \mid f(s,\phi(s,u,y,\lambda),\bar{\lambda}) - f(s,\phi(s,\bar{u},\bar{y},\bar{\lambda}),\bar{\lambda}) \mid ds \end{aligned}$$

For $(s,u,y,\lambda) \in I_{\alpha_0}\times I_{\alpha_0}\times B_\beta \times C_\gamma$, the vectors $(s,\phi(s,u,y,\lambda),\lambda), (s,\phi(s,u,y,\lambda),\bar{\lambda})$ are in $\bar{V}$. Since f is coninuous on $\bar{V}$ and $\bar{V}$ is compact, it is uniformly continuous on $\bar{V}$.

Let $\epsilon > 0$. Then, there is a $\delta > 0$ such that for $\mid \lambda - \bar{\lambda} \mid < \delta$ and any $u \in I_{\alpha_0}, y \in B_\beta$, we have

$$\mid f(u,y,\lambda) - f(u,y,\bar{\lambda}) \mid < \epsilon.$$

Thus, for $\mid \lambda - \bar{\lambda} \mid < \delta$, the first integral in the above inequality is bounded above by $2\alpha\epsilon$.

Thus, for $| \lambda - \bar{\lambda} | < \delta$, and using that f is y-Lipschitz, we have

$$| \phi(t, u, y, \lambda) - \phi(t, \bar{u}, \bar{y}, \bar{\lambda}) |$$

$$\leq \quad | y - \bar{y} | + M| u - \bar{u} | + 2\alpha\epsilon$$

$$+ \int_u^t K| \phi(s, u, y, \lambda) - \phi(s, \bar{u}, \bar{y}, \bar{\lambda}) | ds$$

By the Gronwall inequality, we get

$$| \phi(t, u, y, \lambda) - \phi(t, \bar{u}, \bar{y}, \bar{\lambda}) |$$

$$(| y - \bar{y} | + M| u - \bar{u} | + 2\alpha\epsilon) \, exp(K2\alpha)$$

which in turn gives

$$| \phi(t, u, y, \lambda) - \phi(\bar{t}, \bar{u}, \bar{y}, \bar{\lambda}) |$$

$$\leq (| y - \bar{y} | + M| u - \bar{u} | + 2\alpha\epsilon) \, exp(K2\alpha) + M| t - \bar{t} |.$$

This gives the desired continuity statement. QED

Theorem 2 (Global Continuity of solutions on parameters.) *Suppose the $f(t, x, \lambda)$ is continuous and locally Lipschitz in x in an open set $D \subseteq \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$. If $x(t, a, x_0, \lambda_0)$ is a solution of the IVP $\dot{x} = f(t, x, \lambda_0)$, $x(a) = x_0$ which is defined on the closed interval $[a, b]$ and $(t, x(t, a, x_0, \lambda_0), \lambda_0) \in D$ for $t \in [a, b]$, then there is a neighborhood V of (a, x_0, λ_0) in $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$ such that, for $(u, y, \lambda) \in V$, the IVP $\dot{x} = f(t, x, \lambda)$, $x(u) = y$ also has a solution defined on the interval $[u, b]$. Moreover, the function $x(t, u, y, \lambda)$ is continuous on $[u, b] \times V$.*

Proof. Since $[a, b]$ is a compact set and $x(t, a, x_0, \lambda_0)$ is continuous, the set $A = \{(t, x(t, a, x_0, \lambda_0), \lambda_0) : t \in [a, b]\}$ is a compact subset of D. Therefore, there are constants $M > 0, K > 0$ for which the conditions of the theorem hold with these constants throughout a neighborhood U of A.

Consider the set P of $\beta's$ less than or equal to b in $\mathbf{R}$ for which there is a neighborhood V_β of (a, x_0, λ_0) such that

(*) for $(u, y, \lambda) \in V_\beta$, the IVP $\dot{x} = f(t, x, \lambda)$, $x(u) = y$ has a solution defined on the interval $[u, \beta]$ which is continuous on $[u, \beta] \times V_\beta$.

Then, by the previous theorem, P contains an interval about a (actually of size $\frac{\alpha}{2}$).

Let β_0 be the least upper bound of the set P. If $\beta_0 < b$, the previous result can be used to get a contradiction. QED