3. General Properties of Differential Equations

Let $\mathbf{R}^{n+1}$ be the $n+1-$dimensional Euclidean space and let (t,x) denote coordinates in $\mathbf{R}^{n+1}$ with $x \in \mathbf{R}^n$. Write $\dot{x} = \frac{dx}{dt}$.

A first order ordinary differential equation in $\mathbf{R}^n$ is an expression of the form

$$\dot{x} = f(t,x) \tag{1}$$

where f is a function from an open set $D \subseteq \mathbf{R}^{n+1}$ to $\mathbf{R}^n$. When f depends explicitly on t, the equation (1) is called *non-autonomous* or *time dependent*. If f is independent of t, it is called *autonomous* or *time-independent*.

A solution to (1) is a differentiable function $x(t)$ from a real interval I into $\mathbf{R}^n$ so that

1. $\{(t, x(t)) : t \in I\} \subseteq D$
2. For $t \in I, \dot{x}(t) = f(t, x(t))$.

If we fix a point $(t_0, x_0) \in D$, we are sometimes interested in solutions $x(\cdot)$ of (1) for which $x(t_0) = x_0$.

This leads us to the system of equations

$$\dot{x} = f(t,x), \quad x(t_0) = x_0 \tag{2}$$

which we will call *the initial value problem* of the differential equation (1) with initial value (t_0, x_0) or simply the initial value problem.

Remarks.

1. The $n-$th order scalar differential equation

$$\frac{d^n x}{dt^n} = g(t, x, \dot{x}, \frac{d^2 x}{dt^2}, \ldots, \frac{d^{n-1} x}{dt^{n-1}})$$

can be written as the vector system

$$\begin{aligned} x &= x_1 \\ \frac{dx_1}{dt} &= x_2 \end{aligned}$$

$$\vdots$$
$$\frac{dx_{n-1}}{dt} = x_n$$
$$\frac{dx_n}{dt} = g(t, x_1, \ldots, x_n)$$

using the vector, $(t, x_1, \ldots, x_n)$ with $x_i = \frac{d^{i-1}x}{dt^{i-1}}$ so it is usually not necessary to consider higher order differential equations for general properties.

2. In issues in which $f(t, x)$ is very smooth, e.g. C^∞ it is frequently useful to replace the non-autonomous equation (1) by the system $\dot{t} = 1, \dot{x} = f(t, x)$ and obtain an autonomous equation in one higher dimension.

Examples.

1. The first example shows that even if the right hand side of a differential equation is a polynomial, solutions to (1) may not be defined for all real time.

   Let $D = \mathbf{R}^2, f(t, x) = x^2$. The initial value problem

$$\dot{x} = x^2, x(0) = x_0$$

   has the unique solution $\phi(t) = \frac{-1}{t - x_0^{-1}}$ for $x_0 \neq 0$ and $\phi(t) = 0\ \forall t$ for $x_0 = 0$. For $x_0 \neq 0$, these solutions blow up in finite time.

2. The second example shows that the initial value problem of a continuous differential equation need not have a unique solution.

   Let $D = \mathbf{R}^2, f(t, x) = \sqrt{x}$ for $x \geq 0,\ f(t, x) = 0$ for $x < 0$.

   Fix a real number $c > 0$, and define the function $\phi_c(t) = \frac{(t-c)^2}{4}$ for $t \geq c,\ \phi(t) = 0$ for $t < c$. Then, each $\phi_c(t)$ is a solution to $\dot{x} = f(t, x)$ with value 0 at $t_0 = 0$.

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Lemma. *Suppose that $f(t, x)$ is a continuous function on an open set D in $\mathbf{R}^{n+1}$. Let $(t_0, x_0) \in D$. Then, a continuous function $x(t)$ is a solution to the single integral equation*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

if and only if it is a solution to the initial value problem (2).

Proof.

Suppose that $x(\cdot)$ is a continuous function which solves the integral equation. Then, $x(t_0) = x_0$, and since f is continuous, the Fundamental Theorem of Calculus gives that $x(t)$ is differentiable with

$$\dot{x} = f(t, x(t))$$

so that $x(\cdot)$ solves (2).

Conversely, suppose the $x(\cdot)$ is a solution to the problem (2). Then, $x(\cdot)$ is differentiable, hence continuous, on an interval about t_0. Let $h(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$.

Again the fundamental theorem of calculus gives that h is differentiable with derivative $f(t, x(t)$ at t. Thus, both $x(t)$ and $h(t)$ are differentiable functions with the same derivative on an interval about t_0. Hence, they differ by a constant. But they both have the value x_0 at t_0, so the constant is 0, and $x(t)$ solves the integral equation. QED.

We wish to show that differential equations with continuous right hand sides have solutions at least on small intervals.

Theorem.(Peano Existence Theorem.) *Suppose that $f(t, x)$ is continuous in the open set $D \subseteq \mathbf{R}^{n+1}$. Then, for (t_0, x_0) in D, the initial value problem (2) has at least one solution.*

Proof.

We will give two proofs of this theorem. The first is shorter and depends on a theorem in Functional Analysis.

Definition. Let E be a subset of a Banach space X. The closed convex hull of $E, \bar{co}(E)$ is the intersection of all closed convex sets which contain E. This is clearly the smallest closed convex set containing E.

Theorem.(Mazur) *The closed convex hull of a compact subset E of a Banach space is itself compact.*

Theorem.(Extended Schauder-Tychonov Theorem) *Suppose $\mathcal{A}$ is a closed bounded convex subset of a Banach space and $T : \mathcal{A} \to \mathcal{A}$ is a continuous map such that the image $T\mathcal{A}$ of $\mathcal{A}$ has compact closure. Then, T has a fixed point in $\mathcal{A}$.*

Proof.

Let $B = T\mathcal{A}$. The closure of B is compact, so, by Mazur's theorem, $\bar{co}(B) = \bar{co}(Closure(B)) \equiv B_1$ is also compact.

Since $B \subseteq \mathcal{A}$, we have $closure(B) \subset \mathcal{A}$, since $\mathcal{A}$ is closed and $B_1 \subseteq \mathcal{A}$ since $\mathcal{A}$ is convex. Thus, $TB_1 \subseteq T\mathcal{A} = B \subseteq B_1$, so we may apply the Schauder Theorem to T on B_1 to conclude that T has a fixed point in B_1 which is, of course, also in $\mathcal{A}$. QED

Proof 1 of Peano Theorem:

For $\alpha > 0, \beta > 0$ let $I_\alpha = I_\alpha(t_0) = \{t : | t - t_0 | \leq \alpha\}$ and let $B_\beta = B_\beta(x_0) = \{x : | x - x_0 | \leq \beta\}$.

Choose α, β small enough so that $I_\alpha \times B_\beta \subseteq D$.

Since $I_\alpha \times B_\beta$ is compact and f is continuous on $I_\alpha \times B_\beta$, the quantity

$$M = \sup\{| f(t,x) | : (t,x) \in I_\alpha \times B_\beta\}$$

is finite.

Let α_1 be positive and small enough so that $M\alpha_1 \leq \beta$.

Let $\mathcal{A}$ be the set of continuous functions ϕ from the interval I_{α_1} into $\mathbf{R}^n$ such that

1. $\phi(t_0) = x_0$
2. $| \phi(t) - x_0 | \leq \beta$ for all $t \in I_{\alpha_1}$

Clearly $\mathcal{A}$ is a closed bounded convex subset of the Banach space of coninuous maps from I_{α_1} into $\mathbf{R}^n$ with the sup norm.

Let $T\phi$ be defined by

$$(T\phi)(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds$$

Claim:

1. T maps $\mathcal{A}$ into itself.
2. T is continuous
3. $T\mathcal{A}$ has compact closure.

Once these are established, we have that the extended Schauder-Tychonov Theorem gives us a fixed point ψ of T is $\mathcal{A}$. This fixed point solves the integral equation (), so it provides a solution to the IVP (2).

Proof that T maps $\mathcal{A}$ into itself:

Clearly, $\phi \in \mathcal{A}$ implies that $I_{\alpha_1} \times \phi(I_{\alpha_1}) \subseteq D$ so T is well-defined. Also, $T\phi(t_0) = x_0$.

Next, for $t \in I_{\alpha_1}, \mid \phi(t) - x_0 \mid \leq \mid \int_{t_0}^t f(s, \phi(s))ds \mid \leq M\alpha_1 \leq \beta$, giving statement 1.

Proof of 2. For continuity, suppose that $\epsilon > 0$. We know that f is uniformly continuous on $I_{\alpha_1} \times B_{\beta}$. Let $\delta > 0$ be such that if $\mid (t,x) - (s,y) \mid < \delta$ and $(t,x), (s,y) \in I_{\alpha_1} \times B_{\beta}$, then, $\mid f(t,x) - f(s,y) \mid < \epsilon$.

Now, suppose that $\phi, \psi \in \mathcal{A}$ are such that $\mid \phi - \psi \mid < \delta$. This means that, for each $t \in I_{\alpha_1}, \mid \phi(t) - \psi(t) \mid < \delta$.

Thus, for $t \in I_{\alpha_1}$,

$$\begin{aligned} \mid T\phi(t) - T\psi(t) \mid &\leq \mid \textstyle\int_{t_0}^t f(s, \phi(s)) - f(s, \psi(s))ds \mid \\ &\leq \epsilon \mid t - t_0 \mid \\ &\leq \epsilon\alpha_1 \end{aligned}$$

and it follows that T is continuous on $\mathcal{A}$.

It remains to show that $T\mathcal{A}$ has compact closure. Note that if we show that $T\mathcal{A}$ is equicontinuous, it follows that the closure of $T\mathcal{A}$ is also equicontinuous. Since it is also bounded, it will follow from the Arzela-Ascoli theorem that $T\mathcal{A}$ is compact closure as required.

Equicontinuity of $T\mathcal{A}$:

For $\phi \in \mathcal{A}, t, u \in I_{\alpha_1}$, we have

$$\mid \phi(t) - \phi(u) \mid \leq \mid \textstyle\int_u^t f(s, \phi(s))ds \mid \leq M \mid t - u \mid$$

which gives equicontinuity. QED.

Proof 2 of Peano Theorem.

Let $I_\alpha, I_{\alpha_1}, B_\beta$ be as in Proof 1.

Let $h = h_n = \frac{\alpha_1}{n}$ for $n \geq 1$.

We will consider the Euler polygonal approximations ϕ_h for solutions defined in the following way.

First, let $x_1 = x_0 + f(t_0, x_0)h$. Then, letting $t_{i+1} = t_i + h = t_0 + ih$, set $x_{i+1} = x_i + f(t_i, x_i)h$, for $0 \leq i \leq n - 1$.

This is a discrete sequence of vectors. Interpolate linearly between (t_i, x_i) and (t_{i+1}, x_{i+1}) to form the function

$$\phi_h(t) = x_i + f(t_i, x_i)(t - t_i) \text{ for } t_i \leq t \leq t_{i+1}$$

Claim 1. The sequence of functions $\phi_n = \phi_{h_n}$ is equicontinuous.

First note that they all have the same Lipschitz constant M.

Let $I_j = [t_0 + jh, t_0 + (j+1)h]$. Then, for $s < t$, $s \in I_j, t \in I_l$ with $l \geq j$, we have

$$\begin{aligned} \mid \phi_h(t) - \phi_h(s) \mid \ &\leq \ \mid \phi_h(t) - \phi_h(t_l) \mid + \mid \phi_h(t_l) - \phi_h(t_{l-1}) \mid \\ &\quad + \ldots + \mid \phi_h(t_{j+1}) - \phi_h(s) \mid \\ &\leq \ M(t-s) \end{aligned}$$

Next note that, inductively, if $\mid x_i - x_0 \mid \leq Mih \leq M\alpha_1 < \beta$, then

$$\mid x_{i+1} - x_i \mid \leq Mh$$

so

$$\begin{aligned} \mid x_{i+1} - x_0 \mid \ &\leq \ \mid x_{i+1} - x_i \mid + \mid x_i - x_0 \mid \\ &\leq \ Mh + Mih \\ &\leq \ M(i+1)h \\ &\leq \ M\alpha_1 \end{aligned}$$

for $i < n$.

Similarly, $\mid \phi_{h_n}(t) - x_0 \mid \leq M\alpha_1 \leq \beta$, so the ϕ_{h_n} are uniformly bounded.

Thus, by the Arzela-Ascoli theorem, there is a sequence $\phi_{h_{n_j}}$ which converges to a function $\psi(t)$ of I_{α_1}.

We leave it as an exercise that as $h_n \to 0$,

$$\mid \phi_{h_n}(t) - x_0 - \int_{t_0}^{t} f(s, \phi_{h_n}(s))ds \mid \to 0$$

Thus, for the limit function, ψ, we have

$$\psi(t) = \int_{t_0}^{t} f(s, \psi(s))ds$$

giving us that ψ is a solution. QED.