1 Introduction

This course will cover basic material about ordinary differential equations. The main reference for the initial parts are the First 3 chapters of Hale, Ordinary Differential Equations, 2nd. ed.

Definition. A real normed linear (vector) space is a pair $(\mathcal{X}, |\cdot|)$ where $\mathcal{X}$ is a real vector space and $|\cdot|$: $\mathcal{X} \to \mathbf{R}$ is a real-valued function on $\mathcal{X}$ such that

(i) $| x | \geq 0 \; \forall x$ and $| x | = 0$ iff $x = 0$ for $x \in \mathcal{X}$

(ii) $| \alpha x | = | \alpha || x |$ for $\alpha \in \mathbf{R}, x \in \mathcal{X}$

(iii) $| x + y | \leq | x | + | y | \; \forall \; x, y \in \mathcal{X}$

If $\mathcal{X}$ is a complex vector space and (ii) holds for all $\alpha \in \mathbf{C}$, then we call $(\mathcal{X}, |\cdot|)$ a *complex normed linear space.*

Sometimes we say simply that $\mathcal{X}$ is a normed linear space where we understand that the norm $|\cdot|$ is given implicitly.

If $(\mathcal{X}, |\cdot|)$ is a normed linear space, then the function $d(x, y) = | x - y |$ is a (topological) metric in $\mathcal{X}$.

This means that the pair $(\mathcal{X}, d)$ satisfies the following properties:

(i) $d(x, y) \geq 0$ for all $x, y \in \mathcal{X}$, and $d(x, y) = 0$ iff $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$

As usual, we say that a sequence $(x_1, x_2, \ldots)$ in $(\mathcal{X}, d)$ is a *Cauchy sequence* if, for every $\epsilon > 0$, there is an $N > 0$ such that

$$n, m \geq N \Rightarrow d(x_n, x_m) < \epsilon$$

The metric space $(\mathcal{X}, d)$ is *complete* if every Cauchy sequence in $\mathcal{X}$ converges to a point of $\mathcal{X}$.

Recall that every closed subset of a complete metric space is again complete.

The normed linear space $(\mathcal{X}, | \cdot |)$ is called a *Banach Space* if it is a complete metric space with respect to the metric $d(x, y) = | x - y |$ induced by the norm.

Let us give some examples of normed linear spaces and Banach Spaces.

Examples.

1. Let $\mathcal{X} = R^n$ or $\mathcal{X} = \mathbf{C}^n$ denote the sets of $n-$tuples of real and complex numbers, respectively. Define the following norms $| \cdot |$ in $\mathcal{X}$.

   (a) $| x |_p = (\sum_{1 \leq i \leq n} | x_i |^p)^{\frac{1}{p}}$

   (b) $| x | = \max_{1 \leq i \leq n} | x_i |$

   where $x = (x_1, \ldots, x_n)$ in the above.

   With any one of these norms, $\mathcal{X}$ becomes a Banach space. The *usual* norm is $| \cdot |_2$ above.

   It is instructive to consider the geometric pictures of the unit balls in each of the above Banach Spaces.

2. A linear subspace V of a Banach Space $\mathcal{X}$ is itself a Banach space if and only if it is closed.

3. Let D be a compact subset of $\mathbf{R}^n$. The set $\mathcal{C}(D, \mathbf{R}^n)$ of continuous functions from D to $\mathbf{R}^n$ becomes a Banach Space with the norm

$$| f | = \sup_{x \in D} | f(x) |$$

4. Let X, Y be Banach spaces. Let $B(X, Y)$ be the set of bounded functions from X to Y with the *sup* norm.

$$| f | = \sup_{x \in X} | fx |$$

   Then, $(B(X, Y), | \cdot |)$ is itself Banach space. The set of Bounded continuous functions $BC(X, Y)$ with the sup norm is a closed subspace of $B(X, Y)$.

A function $F : X \to Y$ between metric spaces if called *Lipschitz* if there is a constant $L > 0$ such that

$$d(Fx, Fy) \leq Ld(x, y)$$

for all $x, y \in X$. The smallest such constant,

$$sup_{x \neq y \in X} \frac{d(Fx, Fy)}{d(x, y)}$$

is called the *Lipschitz* constant of F.

Let X, Y be Banach spaces, let $L > 0$, and let $\mathcal{L}_L(X, Y)$ be the set of bounded functions from X to Y which are Lipschitz with Lipschitz constant less than or equal to L. Then, with the *sup* norm, $\mathcal{L}_L(X, Y)$ is a closed subset of $BC(X, Y)$.

Exercises:

1. Suppose that E is a finite dimensional linear vector space. Let $| \cdot |_1, | \cdot |_2$ be two norms on E. Show that there are constants $C_1, C_2 > 0$ such that, for all $x \in E$,

$$C_1 | x |_1 \leq | x |_2 \leq C_2 | x |_1$$

2. Let I be the real unit interval, and let $\mathcal{C}(I, \mathbf{R})$ be the space of continuous real-valued functions on I with the norm

$$| f | = \int_I | f(x) | \, dx$$

Show that this makes $(\mathcal{C}(I, \mathbf{R}), | \cdot |)$ a normed linear space, but that it is not complete. What is the completion of this space?

Definition. Let $\mathcal{X}$ be a compact metric space, and let $\mathcal{F}$ be a collection of continuous functions from $\mathcal{X}$ to the separable Banach space $\mathcal{Y}$ (e.g. $\mathbf{R}^n$). We say that the family $\mathcal{F}$ is *equicontinuous* if, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$x, y \in \mathcal{X} \text{ and } d(x, y) < \delta \Rightarrow | fx - fy | < \epsilon \; \forall \; f \in \mathcal{F}$$

We say the family $\mathcal{F}$ is *bounded* if there is a constant $C > 0$ such that $\mid fx \mid < C$ for all $f \in \mathcal{F}, x \in \mathcal{X}$.

Theorem. (Arzela-Ascoli) *The family $\mathcal{F}$ of functions as above is compact in the uniform topology if and only if it is closed, bounded and equicontinuous.*

Example. Let D be a closed bounded subset of the Euclidean space $\mathbf{R}^n$, and let $\mathcal{Y}$ be a Banach space. Let $K.L > 0$ and let $\mathcal{L}_{L,K}(D, \mathcal{Y})$ be the space of Lipschitz functions from D to $\mathcal{Y}$ with norm less than or equal to K and Lipschitz constant less than or equal to L. Then, $\mathcal{L}_{L,K}(D, \mathcal{Y})$ is a compact metric space. In particular, every sequence in $\mathcal{L}_{L,K}(D, \mathcal{Y})$ has a subsequence which converges to an element of $\mathcal{L}_{L,K}(D, \mathcal{Y})$.