1 Introduction

This course will cover basic material about ordinary differential equations. The main reference for the initial parts are the First 3 chapters of Hale, Ordinary Differential Equations, 2nd. ed.

**Definition.** A real normed linear (vector) space is a pair \((X, | \cdot |)\) where \(X\) is a real vector space and \(| \cdot | : X \to \mathbb{R}\) is a real-valued function on \(X\) such that

(i) \(| x | \geq 0 \forall x \) and \(| x | = 0 \iff x = 0\) for \(x \in X\)

(ii) \(| \alpha x | = | \alpha | | x | \) for \(\alpha \in \mathbb{R}, x \in X\)

(iii) \(| x + y | \leq | x | + | y | \forall x, y \in X\)

If \(X\) is a complex vector space and (iii) holds for all \(\alpha \in \mathbb{C}\), then we call \((X, | \cdot |)\) a **complex normed linear space**.

Sometimes we say simply that \(X\) is a normed linear space where we understand that the norm \(| \cdot |\) is given implicitly.

If \((X, | \cdot |)\) is a normed linear space, then the function \(d(x, y) = | x - y |\) is a (topological) metric in \(X\).

This means that the pair \((X, d)\) satisfies the following properties:

(i) \(d(x, y) \geq 0\) for all \(x, y \in X\), and \(d(x, y) = 0 \iff x = y\)

(ii) \(d(x, y) = d(y, x)\)

(iii) \(d(x, z) \leq d(x, y) + d(y, z)\)

As usual, we say that a sequence \((x_1, x_2, \ldots)\) in \((X, d)\) is a **Cauchy sequence** if, for every \(\epsilon > 0\), there is an \(N > 0\) such that

\[n, m \geq N \Rightarrow d(x_n, x_m) < \epsilon\]

The metric space \((X, d)\) is **complete** if every Cauchy sequence in \(X\) converges to a point of \(X\).

Recall that every closed subset of a complete metric space is again complete.
The normed linear space \((X, | \cdot |)\) is called a **Banach Space** if it is a complete metric space with respect to the metric \(d(x, y) = |x - y|\) induced by the norm.

Let us give some examples of normed linear spaces and Banach Spaces.

**Examples.**

1. Let \(X = \mathbb{R}^n\) or \(X = \mathbb{C}^n\) denote the sets of \(n\)–tuples of real and complex numbers, respectively. Define the following norms \(| \cdot |\) in \(X\).
   
   \[
   (a) \quad |x|_p = \left(\sum_{1 \leq i \leq n} |x_i|^p\right)^{\frac{1}{p}} \\
   (b) \quad |x| = \max_{1 \leq i \leq n} |x_i|
   \]
   
   where \(x = (x_1, \ldots, x_n)\) in the above.

   With any one of these norms, \(X\) becomes a Banach space. The **usual** norm is \(| \cdot |_2\) above.

   It is instructive to consider the geometric pictures of the unit balls in each of the above Banach Spaces.

2. A linear subspace \(V\) of a Banach Space \(X\) is itself a Banach space if and only if it is closed.

3. Let \(D\) be a compact subset of \(\mathbb{R}^n\). The set \(C(D, \mathbb{R}^n)\) of continuous functions from \(D\) to \(\mathbb{R}^n\) becomes a Banach Space with the norm

   \[
   |f| = \sup_{x \in D} |f(x)|
   \]

4. Let \(X, Y\) be Banach spaces. Let \(B(X, Y)\) be the set of bounded functions from \(X\) to \(Y\) with the **sup** norm.

   \[
   |f| = \sup_{x \in X} |fx|
   \]

Then, \((B(X, Y), | \cdot |)\) is itself Banach space. The set of Bounded continuous functions \(BC(X, Y)\) with the sup norm is a closed subspace of \(B(X, Y)\).
A function $F : X \to Y$ between metric spaces if called Lipschitz if there is a constant $L > 0$ such that

$$d(Fx, Fy) \leq Ld(x, y)$$

for all $x, y \in X$. The smallest such constant,

$$\sup_{x \neq y \in X} \frac{d(Fx, Fy)}{d(x, y)}$$

is called the Lipschitz constant of $F$.

Let $X, Y$ be Banach spaces, let $L > 0$, and let $\mathcal{L}_L(X, Y)$ be the set of bounded functions from $X$ to $Y$ which are Lipschitz with Lipschitz constant less than or equal to $L$. Then, with the sup norm, $\mathcal{L}_L(X, Y)$ is a closed subset of $BC(X, Y)$.

**Exercises:**

1. Suppose that $E$ is a finite dimensional linear vector space. Let $\cdot \mid_1, \cdot \mid_2$ be two norms on $E$. Show that there are constants $C_1, C_2 > 0$ such that, for all $x \in E$,

$$C_1 \mid x \mid_1 \leq \mid x \mid_2 \leq C_2 \mid x \mid_1$$

2. Let $I$ be the real unit interval, and let $C(I, \mathbb{R})$ be the space of continuous real-valued functions on $I$ with the norm

$$\mid f \mid = \int_I \mid f(x) \mid dx$$

Show that this makes $(C(I, \mathbb{R}), \mid \cdot \mid)$ a normed linear space, but that it is not complete. What is the completion of this space?

**Definition.** Let $\mathcal{X}$ be a compact metric space, and let $\mathcal{F}$ be a collection of continuous functions from $\mathcal{X}$ to the separable Banach space $Y$ (e.g. $\mathbb{R}^n$). We say that the family $\mathcal{F}$ is **equicontinuous** if, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$x, y \in \mathcal{X} \text{ and } d(x, y) < \delta \Rightarrow \mid fx - fy \mid < \epsilon \ \forall \ f \in \mathcal{F}$$
We say the family $\mathcal{F}$ is bounded if there is a constant $C > 0$ such that $|fx| < C$ for all $f \in \mathcal{F}, x \in \mathcal{X}$.

**Theorem.** (Arzela-Ascoli) The family $\mathcal{F}$ of functions as above is compact in the uniform topology if and only if it is closed, bounded and equicontinuous.

**Example.** Let $D$ be a closed bounded subset of the Euclidean space $\mathbb{R}^n$, and let $\mathcal{Y}$ be a Banach space. Let $K, L > 0$ and let $L_{L,K}(D, \mathcal{Y})$ be the space of Lipschitz functions from $D$ to $\mathcal{Y}$ with norm less than or equal to $K$ and Lipschitz constant less than or equal to $L$. Then, $L_{L,K}(D, \mathcal{Y})$ is a compact metric space. In particular, every sequence in $L_{L,K}(D, \mathcal{Y})$ has a subsequence which converges to an element of $L_{L,K}(D, \mathcal{Y})$. 