Proofs of the Grobman-Hartman theorems - Continued

We now develop the necessary results to prove the Main Step (A) above.

**Lemma 1.** Suppose $H : V \to V$ is a bounded linear self-map of the Banach space $V$ with $|H| < 1$. Let $I$ denote the identity map, $Ix = x$. Then, $I - H$ is an isomorphism and

$$| (I - H)^{-1} | \leq \frac{1}{1 - |H|}$$

(1)

**Proof.**

Let $T = \sum_{i=0}^{\infty} H^i$. Then, $T$ is a bounded linear operator, and

$$(I - H)T = T(I - H) = I.$$  

Therefore, $I - H$ is an isomorphism with inverse $T$.

Moreover,

$$| (I - H)^{-1} | = |T| \leq \sum_{i=0}^{\infty} |H|^i = \frac{1}{1 - |H|} \quad QED.$$  

**Lemma 2.** If $V = V_1 \oplus V_2$ is a direct sum decomposition of the Banach space $V$, and $H : V \to V$ is an isomorphism such that $H(V_i) = V_i$ for $i = 1, 2$, $|H| V_1 | < 1$, and $|H^{-1}| V_2 | < 1$, then $I - H$ is an isomorphism. If $V$ is given the maximum norm, then

$$| (I - H)^{-1} | \leq \max \left( \frac{1}{1 - |H| V_1 |}, \frac{|H^{-1}| V_2 |}{1 - |H^{-1}| V_2 |} \right).$$

(2)

**Proof.**

For $u = u_1 + u_2$ with $u_i \in V_i$, define
\[
T(u) = T(u_1 + u_2) = \sum_{i=0}^{\infty} H^i(u_1) + (-\sum_{i=1}^{\infty} H^{-i}(u_2)).
\]

Then, \((I - H)T = T(I - H) = I\). \(\quad\) QED

**Lemma 3.** Suppose \(L : \mathbb{R}^n \to \mathbb{R}^n\) is a linear map all of whose eigenvalues have norm less than one. Let \(\tau_1 = \sup\{ |\lambda| : \lambda \text{ is an eigenvalue of } L \}\). Let \(\tau \in (\tau_1, 1)\). Then, there is a new norm \(\| \cdot \|\) on \(\mathbb{R}^n\) such that \(\| L(v) \| \leq \tau \| v \|\) for all \(v \in \mathbb{R}^n\). That is, with respect to the norm \(\| \cdot \|\) on \(L\) induced by the norm \(\| \cdot \|\), we have \(\| L \| < \tau\).

**Proof.** Using the fact that \(L = S + N\) where \(S\) is semi-simple (complex diagonalizable) and \(N\) is nilpotent, one sees that there is a constant \(C > 0\) such that \(m \geq 0\) implies that \(\| L^m v \| \leq C(\tau^m) \| v \|\) for all \(v \in \mathbb{R}^n\). Thus, for each \(v\), the quantity \(\alpha(v) = \sup\{ \| L^m v \| \tau^{-m} : m \geq 0 \}\) is finite. Set \(\| v \| = \alpha(v)\). Then, it is easy to see that \(\| \cdot \|\) is a norm on \(\mathbb{R}^n\).

On the other hand,

\[
\| L v \| = \sup\{ \| L^m L v \| \tau^{-m} : m \geq 0 \} \\
= \tau \tau^{-1} \sup\{ \| L^m L v \| \tau^{-m} : m \geq 0 \} \\
= \tau \sup\{ \| L^m v \| \tau^{-m-1} : m \geq 0 \} \\
= \tau \sup\{ \| L^{m+1} v \| \tau^{-m-1} : m \geq 0 \} \\
= \tau \sup\{ \| L^m v \| \tau^{-m} : m \geq 1 \} \\
\leq \tau \| v \|. \quad QED
\]

**Remark.** If we were dealing with a Banach space \(E\) instead of \(\mathbb{R}^n\), we would just let \(\tau_1\) be the spectral radius of the operator \(L\) above.

**Proposition 4.** Suppose \(L : \mathbb{R}^n \to \mathbb{R}^n\) is a linear hyperbolic isomorphism. That is, no eigenvalues of \(L\) have norm 1. Let \(\tau \in (0, 1)\) be such that the eigenvalues of \(L\) inside the unit circle have norm \(< 1\), and those outside the unit circle have norm \(> \tau^{-1}\). Then, there is a direct sum decomposition \(\mathbb{R}^n = V_1 \oplus V_2\) and a new norm \(\| \cdot \|\) on \(\mathbb{R}^n\) such that
\[
L(V_1) = V_1, \quad L(V_2) = V_2
\]  

and

\[
\| L \|_{V_1} < \tau, \quad \| L^{-1} \|_{V_2} < \tau
\]

Proof. Let \( \mathbb{R}^n = V_1 \oplus V_2 \) be the direct sum decomposition such that \( L \| V_1 \) has eigenvalues less than \( \tau \) in norm, and \( L \| V_2 \) has eigenvalues greater than \( \tau^{-1} \) in norm. Note that \( L^{-1} \| V_2 \) has eigenvalues of norm less than \( \tau \). By Lemma 3, there are norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) on \( V_1 \) and \( V_2 \), respectively, such that (4) holds. For \( v = (v_1, v_2) \) with \( v_i \in V_i \), let \( \| v \| = \max(\| v_1 \|, \| v_2 \|) \).

QED

Proof of Main Step (A).

We show that the equation

\[
(L + \phi_1) \circ (id + u_1) = (id + u_1) \circ (L + \phi_2) \quad \text{with Lip}(\phi_i) < \varepsilon
\]

has a unique solution \( u_1 \in C^0_b(\mathbb{R}^n, \mathbb{R}^n) \) for \( \varepsilon \) small.

Equation (5) is equivalent to

\[
L \circ id + L \circ u_1 + \phi_1 \circ (id + u_1) = L + \phi_2 + u_1 \circ (L + \phi_2)
\]

or,

\[
u_1 - L^{-1} u_1 \circ (L + \phi_2) = L^{-1} \phi_2 - L^{-1} \phi_1 \circ (id + u_1).
\]

Let \( H : C^0_b(\mathbb{R}^n, \mathbb{R}^n) \to C^0_b(\mathbb{R}^n, \mathbb{R}^n) \) be defined by

\[
H(u) = L^{-1} \circ u \circ (L + \phi_2),
\]

and let \( H_1 = I - H \) with \( I \) the identity transformation of \( C^0_b(\mathbb{R}^n, \mathbb{R}^n) \). Then, both \( H \) and \( H_1 \) are bounded linear maps, and equation (6) becomes

\[
H_1(u_1) = L^{-1} \phi_2 - L^{-1} \phi_1 \circ (id + u_1).
\]
Claim (B): \( H_1 \) is an isomorphism and \( |H_1^{-1}| \leq \frac{1}{1 - \tau} \).

**Exercise.** (Lipschitz Inverse Function Theorem). Let \((V, |\cdot|)\) be a Banach space, and suppose \( f : V \to V \) is 1-1, onto, and Lipschitz with Lipschitz inverse. There is an \( \varepsilon > 0 \) such that if \( g = f + \phi \) where \( \phi \) is Lipschitz with \( ||\phi||_0 < \varepsilon \) and \( \text{Lip}(\phi) < \varepsilon \), then \( g \) is 1-1, onto, and Lipschitz with Lipschitz inverse.

**Proof of Claim (B).** Note that by the exercise, for \( \varepsilon \) small, \((L + \phi)^{-1}\) exists and is Lipschitz. This gives that \( H \) is an isomorphism with inverse \( u \to L \circ u \circ (L + \phi)^{-1} \).

Let \( \bar{V}_i = C^0_b(\mathbb{R}^n, V_i) \) for \( i = 1, 2 \). Then, \( C^0_b(\mathbb{R}^n, \mathbb{R}^n) = \bar{V}_1 \oplus \bar{V}_2 \), \( H(\bar{V}_i) = \bar{V}_i \), \( |H| |\bar{V}_1| < \tau \), and \( |H^{-1}| |\bar{V}_1| < \tau \). Thus, \( H \) is hyperbolic on \( C^0_b(\mathbb{R}^n, \mathbb{R}^n) \). By Lemma 2, we have that \( H_1 \) is an isomorphism and \( |H_1^{-1}| \leq \frac{1}{1 - \tau} \) which is Claim (B).

Now, (6) becomes
\[
H_1(u_1) = L^{-1}\phi_2 - L^{-1}\phi_1 \circ (id + u_1)
\]
or
\[
u_1 = H_1^{-1}(L^{-1}\phi_2 - L^{-1}\phi_1 \circ (id + u_1)) = H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id + u_1))
\]
which means we want a fixed point in \( C^0_b(\mathbb{R}^n, \mathbb{R}^n) \) of the map
\[
T : u \to H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id + u))
\]
We show that \( T \) is a contraction if \( \varepsilon \) is small.

We have,
\[
|T u - T v|_0 = ||H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id + u)) - (H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id + v)))||_0
\]
\[
= ||H_1^{-1}(L^{-1}\phi_1 \circ (id + v)) - H_1^{-1}(L^{-1}\phi_1 \circ (id + u))||_0
\]
\[
\leq \|H_1^{-1}\|_0 \|L^{-1}\||\phi_1 \circ (id + u) - \phi_1 \circ (id + v)||_0
\]
\[
\leq \|H_1^{-1}\|_0 \|L^{-1}\||(\text{Lip}(\phi_1))||u - v||_0.
\]
So, if

\[ \text{Lip}(\phi_1) | L^{-1} | \frac{1}{1 - \tau} < 1, \]

then \( T \) is a contraction.

This completes the proofs of Theorems 1 and 2.