Proofs of the Grobman-Hartman theorems - Continued

We now develop the necessary results to prove the Main Step (A) above.

Lemma 1. Suppose $H : V \to V$ is a bounded linear self-map of the Banach space V with $| H | < 1$. Let I denote the identity map, $Ix = x$. Then, $I - H$ is an isomorphism and

$$| (I-H)^{-1} | \leq \frac{1}{1 - | H |} \tag{1}$$

Proof.

Let $T = \sum_{i=0}^{\infty} H^i$. Then, T is a bounded linear operator, and

$$(I-H)T = T(I-H) = I.$$

Therefore, $I - H$ is an isomorphism with inverse T.
Moreover,

$$| (I-H)^{-1} | = | T | \leq \sum_{i=0}^{\infty} | H |^i = \frac{1}{1 - | H |} \quad QED.$$

Lemma 2. *If $V = V_1 \oplus V_2$ is a direct sum decomposition of the Banach space V, and $H : V \to V$ is an isomorphism such that $H(V_i) = V_i$ for $i = 1, 2$, $| H \mid V_1 | < 1$, and $| H^{-1} \mid V_2 | < 1$, then $I - H$ is an isomorphism. If V is given the maximum norm, then*

$$| (I-H)^{-1} | \leq \max \left(\frac{1}{1 - | H \mid V_1 |}, \frac{| H^{-1} \mid V_2 |}{1 - | H^{-1} \mid V_2 |} \right). \tag{2}$$

Proof.

For $u = u_1 + u_2$ with $u_i \in V_i$, define

$$T(u) = T(u_1 + u_2) = \sum_{i=0}^{\infty} H^i(u_1) + (-\sum_{i=1}^{\infty} H^{-i}(u_2)).$$

Then, $(I - H)T = T(I - H) = I$. QED

Lemma 3. *Suppose $L : \mathbf{R}^n \to \mathbf{R}^n$ is a linear map all of whose eigenvalues have norm less than one. Let $\tau_1 = \sup\{| \lambda | : \lambda$ is an eigenvalue of $L\}$. Let $\tau \in (\tau_1, 1)$. Then, there is a new norm $|| \cdot ||$ on $\mathbf{R}^n$ such that $|| L(v) || \leq \tau || v ||$ for all $v \in \mathbf{R}^n$. That is, with respect to the norm $||| \cdot |||$ on L induced by the norm $|| \cdot ||$, we have $||| L ||| < \tau$.*

Proof. Using the fact that $L = S + N$ where S is semi-simple (complex diagonalizable) and N is nilpotent, one sees that there is a constant $C > 0$ such that $m \geq 0$ implies that $| L^m v | \leq C(\tau^m)| v |$ for all $v \in \mathbf{R}^n$. Thus, for each v, the quantity $\alpha(v) = \sup(\{| L^m v |\tau^{-m} : m \geq 0\})$ is finite. Set $|| v || = \alpha(v)$. Then, it is easy to see that $|| \cdot ||$ is a norm on $\mathbf{R}^n$.

On the other hand,

$$\begin{aligned}
|| Lv || &= \sup(\{| L^m Lv |\tau^{-m} : m \geq 0\}) \\
&= \tau\tau^{-1} \sup(\{| L^m Lv |\tau^{-m} : m \geq 0\}) \\
&= \tau \sup(\{| L^m Lv |\tau^{-m-1} : m \geq 0\}) \\
&= \tau \sup(\{| L^{m+1} v |\tau^{-m-1} : m \geq 0\}) \\
&= \tau \sup(\{| L^m v |\tau^{-m} : m \geq 1\}) \\
&\leq \tau || v ||. \quad QED
\end{aligned}$$

Remark. If we were dealing with a Banach space E instead of $\mathbf{R}^n$, we would just let τ_1 be the spectral radius of the operator L above.

Proposition 4. *Suppose $L : \mathbf{R}^n \to \mathbf{R}^n$ is a linear hyperbolic isomorphism. That is, no eigenvalues of L have norm 1. Let $\tau \in (0, 1)$ be such that the eigenvalues of L inside the unit circle have norm < 1, and those outside the unit circle have norm $> \tau^{-1}$. Then, there is a direct sum decomposition $\mathbf{R}^n = V_1 \oplus V_2$ and a new norm $|| \cdot ||$ on $\mathbf{R}^n$ such that*

$$L(V_1) = V_1, \;\; L(V_2) = V_2 \tag{3}$$

and

$$|| \, L \mid V_1 \, || < \tau, \;\; || \, L^{-1} \mid V_2 \, || < \tau \tag{4}$$

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Proof. Let $\mathbf{R}^n = V_1 \oplus V_2$ be the direct sum decomposition such that $L \mid V_1$ has eigenvalues less than τ in norm, and $L \mid V_2$ has eigenvalues greater than τ^{-1} in norm. Note that $L^{-1} \mid V_2$ has eigenvalues of norm less than τ. By Lemma 3, there are norms $|| \cdot ||_1$ and $|| \cdot ||_2$ on V_1 and V_2, respectively, such that (4) holds. For $v = (v_1, v_2)$ with $v_i \in V_i$, let $|| \, v \, || = \max(|| \, v_1 \, ||, || \, v_2 \, ||)$.
QED

Proof of Main Step (A)**.**

We show that the equation

$$(L + \phi_1) \circ (id + u_1) = (id + u_1) \circ (L + \phi_2) \text{ with } Lip(\phi_i) < \varepsilon \tag{5}$$

has a unique solution $u_1 \in C_b^0(\mathbf{R}^n, \mathbf{R}^n)$ for ε small.

Equation (5) is equivalent to

$$L \circ id + L \circ u_1 + \phi_1 \circ (id + u_1) = L + \phi_2 + u_1 \circ (L + \phi_2)$$

or,

$$u_1 - L^{-1} u_1 \circ (L + \phi_2) = L^{-1}\phi_2 - L^{-1}\phi_1 \circ (id + u_1). \tag{6}$$

Let $H : C_b^0(\mathbf{R}^n, \mathbf{R}^n) \to C_b^0(\mathbf{R}^n, \mathbf{R}^n)$ be defined by

$$H(u) = L^{-1} \circ u \circ (L + \phi_2),$$

and let $H_1 = I - H$ with I the identity transformation of $C_b^0(\mathbf{R}^n, \mathbf{R}^n)$. Then, both H and H_1 are bounded linear maps, and equation (6) becomes

$$H_1(u_1) = L^{-1}\phi_2 - L^{-1}\phi_1 \circ (id + u_1). \tag{7}$$

$$\text{Claim } (B): \qquad H_1 \text{ is an isomorphism and } \mid H_1^{-1} \mid \leq \frac{1}{(1-\tau)}.$$

Exercise. (Lipschitz Inverse Function Theorem). *Let* $(V, |\cdot|)$ *be a Banach space, and suppose* $f : V \to V$ *is 1-1, onto, and Lipschitz with Lipschitz inverse. There is an* $\varepsilon > 0$ *such that if* $g = f + \phi$ *where* ϕ *is Lipschitz with* $\| \phi \|_0 < \varepsilon$ *and* $Lip(\phi) < \varepsilon$*, then* g *is 1-1, onto, and Lipschitz with Lipschitz inverse.*

Proof of Claim (B)**.** Note that by the exercise, for ε small, $(L + \phi_2)^{-1}$ exists and is Lipschitz. This gives that H is an isomorphism with inverse $u \to L \circ u \circ (L + \phi_2)^{-1}$.

Let $\bar{V}_i = C_b^0(\mathbf{R}^n, V_i)$ for $i = 1, 2$. Then, $C_b^0(\mathbf{R}^n, \mathbf{R}^n) = \bar{V}_1 \oplus \bar{V}_2$, $H(\bar{V}_i) = \bar{V}_i$, $\mid H \mid \bar{V}_2 \mid < \tau$, and$\mid H^{-1} \mid \bar{V}_1 \mid < \tau$. Thus, H is hyperbolic on $C_b^0(\mathbf{R}^n, \mathbf{R}^n)$. By Lemma 2, we have that H_1 is an isomorphism and $\mid H_1^{-1} \mid \leq \frac{1}{1-\tau}$ which is Claim (B).

Now, (6) becomes

$$H_1(u_1) = L^{-1}\phi_2 - L^{-1}\phi_1 \circ (id + u_1)$$

or

$$u_1 = H_1^{-1}(L^{-1}\phi_2 - L^{-1}\phi_1 \circ (id + u_1)) = H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id + u_1))$$

which means we want a fixed point in $C_b^0(\mathbf{R}^n, \mathbf{R}^n)$ of the map

$$T : u \to H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id + u))$$

We show that T is a contraction if ε is small.

We have,

$$\begin{aligned}
\| Tu - Tv \|_0 &= \| H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id + u)) \\
&\quad -(H_1^{-1}(L^{-1}\phi_2) - H_1^{-1}(L^{-1}\phi_1 \circ (id + v))) \|_0 \\
&= \| H_1^{-1}(L^{-1}\phi_1 \circ (id + v)) - H_1^{-1}(L^{-1}\phi_1 \circ (id + u)) \|_0 \\
&\leq \mid H_1^{-1} \|\, L^{-1} \|\|\, \phi_1 \circ (id + u) - \phi_1 \circ (id + v) \|_0 \\
&\leq \mid H_1^{-1} \|\, L^{-1} \mid (Lip(\phi_1)) \|\, u - v \|_0.
\end{aligned}$$

So, if

$$Lip(\phi_1)|\ L^{-1}\ |\frac{1}{1-\tau} < 1,$$

then T is a contraction.

This completes the proofs of Theorems 1 and 2.